ON SOME PROBLEMS OF INTERPOLATION AND APPROXIMATION THEORY

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1992

To my Parents

Frederick and Catherine Burkett

ACKNOWLEDGMENTS

The author wishes to express sincere thanks to Dr. Arun Kumar Varma for his many hours of discussions and counseling in the field of approximation theory. His contagious enthusiasm for the subject and his desire to share this enthusiasm made this dissertation possible. The author is grateful for the generous advice and comments made by the members of the Ph.D. committee. They are Dr. Edwards, Dr. Mair, Dr. Popov, and Dr. Sheppard, in addition to Dr. Varma. Special thanks belong to Dr. Szabodos for his kindness and suggestions during a visit to the University of Florida. Finally, the author thanks Dr. Drake and the Department of Mathematics for their support.

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Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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December 1992

Chairman: Dr. Arun K. Varma
Major Department: Mathematics

We start by presenting some well known results concerning polynomial approximation, Birkhoff interpolation, lacunary spline interpolation, and Markov type inequalities. These results provide a historical motivation for the problems considered in later chapters.

After this introduction, we give the explicit representation and convergence properties for a Birkhoff interpolation process. We find the unique algebraic polynomial that takes on function values for a specific set of knots. In addition, the second derivative of this polynomial takes the value zero on another set of prescribed knots. Next, we look at how well these polynomials approximate a given function. It is shown that the polynomials converge uniformly for any continuous function on the closed interval. In fact, we present a pointwise

estimate which provides a discrete, interpolatory proof of the Teljakovskii Theorem of Approximation.

Next, motivated by earlier results of Meir and Sharma, we consider lacunary spline interpolation in the (0,1,3) and (0,1,2,4) cases. For (0,1,3) interpolation, function values and third derivative values are prescribed at the joints, while function values and first derivative values are prescribed at the midpoints of the joints. Similarly, we consider the problem of (0,1,2,4) interpolation. One such spline turns out to be local in character. Specifically, it is determined by the solution of a diagonal matrix.

Motivated by an open problem of P. Turán, Rahman studied the extremal properties of polynomials under curved majorants in the uniform norm. We discuss similar results in the L^p norm. We present theorems for both the circular and parabolic majorants.

CHAPTER ONE
INTRODUCTION

Polynomial Approximation

In 1885, Weierstrass showed that an arbitrary continuous function on a compact interval can be uniformly approximated by a sequence of polynomials. We more precisely state this as follows.

Let $f \in C[-1,1]$. Here $C[-1,1]$ denotes the class of functions continuous on $[-1,1]$. For any $\epsilon > 0$, there exists a polynomial $P(x)$ such that $\|f(x) - P(x)\| < \epsilon$. Here, we denote the usual sup norm,

$$\|f\| = \sup_{-1 \le x \le 1} |f(x)|.$$

In 1909, Dunham Jackson chose, as a thesis topic, to investigate the degree of approximation with which a given continuous function can be represented by a polynomial of given degree [21]. This gave rise to the Jackson Theorem.

Before presenting the theorem, we define the concept of best approximation for a given function. Let $f \in C[-1,1]$. Then

$$E_n(f) = \inf_{P_n \in \pi_n} \|f - P_n\| \tag{1.1.1}$$

where π_n represents the class of algebraic polynomials of degree $\leq n$.

Theorem 1.1 (Jackson). Let $f \in C[-1,1]$. There exists a positive constant A such that

$$(1.1.2) \qquad E_n(f) \leq A\omega(f, \frac{1}{n}) \text{ for } n = 1, 2, \ldots$$

where A is independent of f.

Note: $\omega(f,\delta)$ represents the usual modulus of continuity defined by

$$\omega(f, \delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|.$$

A more rapid decrease to zero for $E_n(f)$ is possible if we assume more smoothness for f. Dunham Jackson proved the following result on differentiable functions.

Let $C^r[-1,1]$ denote the class of functions f such that $f^{(r)} \in C[-1,1]$.

Theorem 1.2 (Jackson). If $f \in C^r[-1,1]$, then

$$(1.1.3) \qquad E_n(f) \leq A_r \left(\frac{1}{n}\right)^r \omega\left(f^{(r)}, \frac{1}{n}\right) \text{ for } n = 1, 2, \ldots$$

where A_r is a constant independent of f.

In 1951, A.F. Timan obtained the following improvement of the Jackson Theorem.

Theorem 1.3 (Timan). Let $f \in C[-1,1]$. There exists a positive constant B and a polynomial P_n of degree n such that

$$(1.1.4) \qquad |f(x) - P_n(x)| \le B\left[\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right) + \omega\left(f, \frac{1}{n^2}\right)\right]$$

for $-1 \le x \le 1$ and $n = 1, 2, \ldots$ where B is independent of f.

Timan's theorem gives a pointwise estimate for $|f(x) - P_n(x)|$. Notice that as $|x|$ approaches 1, the order of convergence near the ends is better than at the middle of $[-1,1]$.

We now state a needed definition. Let Lip α represent the class of functions f in $C[-1,1]$ such that

$$(1.1.5) \qquad |f(x) - f(y)| \le M|x-y|^{\alpha} \text{ for all x and y in } [-1, 1]$$

where M is some fixed constant.

We now state a converse to the Timan Theorem proved by V.K. Dzjadyk in 1956.

Theorem 1.4 (Dzjadyk). Let $f \in C[-1,1]$ and $0 < \alpha < 1$. There exists a constant B and a polynomial P_n of degree n such that

$$(1.1.6) \qquad |f(x) - P_n(x)| \le B\left[\left(\frac{\sqrt{1-x^2}}{n}\right)^{\alpha} + \left(\frac{1}{n^2}\right)^{\alpha}\right] \text{ for } -1 \le x \le 1$$

and $n = 1, 2, \ldots$ if and only if $\omega(f,h) \le Ch^{\alpha}$ for some constant C.

A further improvement of the Jackson and Timan Theorems was made by Teljakovskii in 1966. Here, it was shown that the estimate of $|f(x) - P_n(x)|$ can be made exact at the endpoints of the interval.

Theorem 1.5 (Teljakovskii). Let $f \in C\,[-1, 1]$. There exists a positive constant D and a polynomial P_n of degree n such that

$$(1.1.7) \qquad |f(x) - P_n(x)| \le D\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right) \text{ for } -1 \le x \le 1 \text{ and}$$

n = 1, 2, ... where D is independent of f.

Modifying the Jackson operator, Ron DeVore strengthened the above theorem as follows [10].

For a function f(x) bounded on [-1,1], we define the second modulus of smoothness by

$$(1.1.8) \qquad \omega_2(f,\delta) = \sup_{-1 \le x,\, x+2h \le 1,\, |h| \le \delta} |f(x) - 2f(x+h) + f(x+2h)|.$$

Theorem 1.6 (DeVore). Let $f \in C[-1,1]$. There exists a positive constant A and a polynomial P_n of degree n such that

$$(1.1.9) \qquad |f(x) - P_n(x)| \le A\omega_2\left(f, \frac{\sqrt{1-x^2}}{n}\right) \text{ for } -1 \le x \le 1 \text{ and}$$

n = 1, 2, ... where A is independent of f.

There are a number of proofs for the Teljakovskii and DeVore Theorems. The initial proofs involved convolution of the approximated function with the Jackson kernel. This

requires the function to be known almost everywhere. In 1979, Mills and Varma obtained a discrete, weakly interpolatory proof of the Teljakovskii Theorem [24]. In 1989, Varma and Yu obtained such a proof for the DeVore Theorem [48]. These proofs require the function values to be known at only a discrete number of points.

This concludes our brief discussion on polynomial approximation. The reader may obtain other important contributions from the book of Timan [39].

Lagrange and Hermite - Fejér Interpolation

Let X denote an infinite triangular matrix with all entries in [-1,1]

$$(1.2.1)\qquad X : \begin{array}{llll} x_{0,0} & & & \\ x_{0,1} & x_{1,1} & & \\ \vdots & & & \\ x_{0,n} & x_{1,n} & \cdots & x_{n,n} \\ \cdot & & & \end{array}$$

We denote by $L_n[f,x;X]$, the Lagrange polynomial of degree $\leq$ n which interpolates f(x) at the nodes $x_{k,n}$ for k = 0, 1, ..., n. Then

$$(1.2.2)\qquad L_n[f,x;X] = \sum_{k=0}^{n} f(x_{k,n})\, l_{k,n}(x)$$

where

$$l_{k,n}(x) = \frac{\omega_n(x)}{(x-x_{k,n})\,\omega_n'(x_{k,n})},\quad \omega_n(x) = \prod_{k=0}^{n}(x-x_{k,n})\,.$$

For a time, it was thought that for some matrix X, the Lagrange interpolating polynomials converge uniformly to any given continuous function on [-1,1]. The hopes for this idea vanished when Bernstein and Faber simultaneously discovered in 1914 that for any triangular system of interpolation points, we can construct a continuous function for which the Lagrange interpolatory process carried out on these points cannot converge uniformly to this function.

In 1916, L. Fejér showed that if instead of Lagrange interpolation we consider Hermite-Fejér interpolation the situation changes [17]. The Hermite-Fejér polynomials $H_n[f,x;X]$ are of degree $\leq 2n + 1$ and uniquely determined by the conditions

$$(1.2.3) \qquad H_n[f,x_{k,n};X] = f(x_{k,n}),\ H_n'[f,x_{k,n};X] = 0$$

for k = 0, 1, ..., n. Fejér showed that for particular matrices X, as in (1.2.1), the Hermite-Fejér interpolating polynomials converge uniformly to any given function $f \in C[-1,1]$. For example, choosing the knots to be the zeros of the Tchebycheff polynomial $T_n(x) = \cos n\theta$, $x = \cos\theta$ guarantees convergence for the entire class of continuous functions on the closed interval [-1,1].

Birkhoff Interpolation

In problems of Hermite type interpolation, function values and consecutive derivative values are prescribed at

given points. In 1906, G.D. Birkhoff considered those interpolation problems in which the consecutive derivative requirement can be dropped [6]. This more general kind of interpolation is now referred to as Birkhoff (or lacunary) interpolation.

The problems of Birkhoff interpolation differ greatly from Lagrange and Hermite interpolation. For example, Lagrange and Hermite interpolation problems are always uniquely solvable for a given set of knots, but a given problem in Birkhoff interpolation may not have a unique solution.

More precisely, given n + 1 integer pairs (i,k) corresponding to n + 1 real numbers $y_{i,k}$ and m distinct real numbers x_i, i = 1, 2, ..., m $\leq$ n + 1, a given problem of polynomial interpolation is to satisfy the n + 1 equations

$$(1.3.1) \qquad P_n^{(k)}(x_i) = y_{i,k}$$

with a polynomial P_n of degree at most n. (We use the convention $P_n^{(o)}(x) = P_n(x)$.) For each i, the orders k of the derivatives in (1.3.1) form a sequence k = 0, 1, ..., k_i. If one or more of the sequences is broken, we have Birkhoff interpolation.

A number of different cases in Birkhoff interpolation have been studied. In its first general treatment, Turán and associates studied (0,2) interpolation where the knots

are the zeros of the integral of the Legendre polynomial [3][4][34]. It was found that these interpolating polynomials exist uniquely only when the number of knots used is even. We state this result as a theorem.

Define

$$(1.3.2) \qquad \pi_n(x) = (1-x^2)\, P'_{n-1}(x)$$

where $P_{n-1}(x)$ is the Legendre polynomial of degree $n - 1$ normalized by $P_{n-1}(1) = 1$. An equivalent definition of $\pi_n(x)$ is

$$(1.3.3) \qquad \pi_n(x) = -n(n-1) \int_{-1}^{x} P_{n-1}(t)\, dt.$$

<u>Theorem 1.7</u> (Turán and Suranyi). Given arbitrary real values $(a_{1,n}, a_{2,n}, \ldots, a_{n,n})$ and $(b_{1,n}, b_{2,n}, \ldots, b_{n,n})$ where n is an even positive integer, there exists a unique real algebraic polynomial $R_n(x)$ of degree $\leq 2n - 1$ such that

$$(1.3.4) \qquad R_n(x_{i,n}) = a_{i,n} \text{ and } R''_n(x_{i,n}) = b_{i,n}$$

for $i = 1, 2, \ldots, n$ where

$$-1 = x_{1,n} < x_{2,n} < \ldots < x_{n,n} = 1$$

are the zeros of $(1 - x^2)P'_{n-1}(x)$.

Later, Varma and Prasád proved the following [47].

Theorem 1.8 (Varma and Prasád). Given arbitrary real values $(c_{1,n}, c_{2,n}, \ldots, c_{n,n})$ and $(d_{2,n}, d_{3,n}, \ldots, d_{n-1,n})$ where n is an even positive integer, there exists a unique real algebraic polynomial $Q_n(x)$ of degree $\leq 2n - 3$ such that

$$(1.3.5) \qquad Q_n(x_{i,n}) = c_{i,n} \text{ for } i = 1, 2, \ldots, n \text{ and}$$
$$Q_n''(x_{i,n}) = d_{i,n} \text{ for } i = 2, 3, \ldots, n-1$$

where

$$-1 = x_{1,n} < x_{2,n} < \ldots < x_{n,n} = 1$$

are the zeros of $(1 - x^2)\, P_{n-2}(x)$.

After answering the questions of existence and uniqueness, it is natural to address the problem of convergence. Turán and Balázs followed Theorem 1.6 with a result on convergence which was subsequently improved by Freud [18].

Theorem 1.9 (Turán and Balázs, improved by Freud). Let $f \in C[-1,1]$ such that $\omega_2(f,h) = h\,\epsilon(h)$ where $\epsilon(h) \to 0$ as $h \to 0$.

If $|\beta_{i,n}| \leq \delta_n$ where $n^{-1}\,\delta_n \to 0$, then

$$(1.3.6) \qquad R_n(f,x) \to f(x) \text{ uniformly for } -1 \leq x \leq 1.$$

Here, $R_n(f,x)$ is given by

$$(1.3.7) \qquad R_n(f,x) = \sum_{k=1}^{n} f(x_{k,n})\, r_{k,n}(x) + \sum_{k=1}^{n} \beta_{k,n}\rho_{k,n}(x)$$

where the explicit forms of $r_{k,n}(x)$ and $\rho_{k,n}(x)$ are given by Balázs and Turán [3].

Other cases of (0,2) interpolation have also been studied. For example, Varma studied the convergence properties of the (0,2) interpolating polynomials where the knots are the zeros of Tchebycheff polynomials of the first kind [41]. Here, it has been shown by Surányi and Turán, the polynomials exist uniquely for an even number of knots [34].

<u>Theorem 1.10</u> (Varma). Let $f \in C^1[-1,1]$ and let $f' \in \text{Lip } \alpha$, $\alpha > \frac{1}{2}$.

If

$$(1.3.8) \qquad \delta_{i,n} = \frac{\epsilon_n\sqrt{n}}{(1-x_{i,n}^2)} \quad \text{for } i = 2, 3, \ldots , n+1$$

where $x_{i,n}$ are the zeros of $T_n(x) = \cos n\theta$, $x = \cos\theta$ and

$$(1.3.9) \qquad \lim_{n\to\infty} \epsilon_n = 0 .$$

Then $S_n(f,x)$ converges uniformly to $f(x)$ in $[-1,1]$ where

$$(1.3.10) \quad S_n(f,x) = \sum_{i=2}^{n+1} f(x_{i,n})\, u_{i,n}(x) + \sum_{i=2}^{n+1} \delta_{i,n}\, v_{i,n}(x)\,.$$

Here $u_{i,n}(x)$ and $v_{i,n}(x)$ are given by Varma [41].

Saxena and Sharma extended (0,2) interpolation to the case of (0,1,3) interpolation [33]. They considered the problems of existence and uniqueness of polynomials which interpolate prescribed function values, first derivative values, and third derivative values at a given set of points. They also obtained convergence results analogous to Theorem 1.9.

In 1989, Akhlaghi, Chak, and Sharma addressed the problem of (0,3) interpolation based on the zeros of $\pi_n(x)$ [1]. They found the (0,3) interpolating polynomials to exist uniquely for every $n \geq 4$. In addition, explicit forms were found for the fundamental polynomials though complicated in nature.

Subsequently, Szabodos and Varma [36] found simpler explicit forms for this (0,3) interpolation and, consequently, were able to obtain the following convergence result.

<u>Theorem 1.11</u> (Szabodos and Varma). Let $f \in C[-1,1]$. Then

$$(1.3.11) \quad \| f(x) - R_n(f,x) \| = O\left(\omega_3\left(f, \frac{\log^{1/3} n}{n}\right)\right)$$

where $\omega_3(f,h)$ is the modulus of smoothness of order 3 of $f(x)$. Here

$$(1.3.12) \quad R_n(f,x) = \sum_{j=1}^{n} f(x_{j,n})\, r_{j,n}(x)$$

where $r_{j,n}(x)$ are the fundamental polynomials of the first kind.

Notice that these (0,3) interpolating polynomials converge uniformly for a wider class of functions than in the (0,2) interpolation theorems we presented. In fact, there is an open problem of Turán to find the "most stable" (0,2) interpolation in the following sense [40].

Problem XXXI.

Given a matrix X as in (1.2.1) such that each row contains knots where the (0,2) interpolating polynomials exist, find the matrix that will minimize

$$(1.3.13) \quad \max_{-1\le x\le 1} \sum_{k=1}^{n} |r_{k,n}(x)|$$

where $r_{k,n}(x)$ are the fundamental polynomials of the first kind.

Since $r_{k,n}(x_{k,n}) = 1$, we cannot hope to do better than

$$(1.3.14) \quad \max_{-1\le x\le 1} \sum_{k=1}^{n} |r_{k,n}(x)| = O(1) \text{ for some matrix x.}$$

Although (1.3.14) has not been obtained in the strictest sense, such a result has been recently obtained by the use of two different sets of knots. This type of

interpolation has been referred to as Pál Type interpolation.

Akhlaghi and Sharma have studied (0,2) interpolation on two different sets of knots, namely the zeros of $(1 - x^2)$ $P'_{n-1}(x)$ and $P_{n-1}(x)$ [2]. They established that these interpolating polynomials exist uniquely for n even or odd. In addition, some results on explicit forms were obtained which we will not state here. We do, however, present the following.

<u>Theorem 1.12</u> (Akhlaghi and Sharma). Given arbitrary real values $(a_{1,n}, a_{2,n}, \ldots , a_{n,n})$ and $(b_{1,n}, b_{2,n}, \ldots , b_{n-1,n})$, there exist unique real algebraic polynomials $S_n(x)$ and $T_n(x)$ of degree $\leq 2n - 2$ each such that

(1.3.15) $S_n(x_{i,n}) = a_{i,n}$ for $i = 1, 2, \ldots , n$ and

$S''_n(y_{i,n}) = b_{i,n}$ for $i = 1, 2, \ldots , n - 1$ and

(1.3.16) $T_n(y_{i,n}) = b_{i,n}$ for $i = 1, 2, \ldots , n - 1$ and

$T''_n(x_{i,n}) = a_{i,n}$ for $i = 1, 2, \ldots , n$.

Here, $x_{i,n}$ are the zeros of $(1 - x^2)$ $P'_{n-1}(x)$ and $y_{i,n}$ are the zeros of $P_{n-1}(x)$.

Subsequently, Szabodos and Varma proved uniqueness and existence for a modified (0,2) process very similar to the one described in (1.3.15). Their modified (0,2) process differs in that it also prescribes first derivative values

at the endpoints ± 1. Before presenting the convergence results, we define

$$(1.3.17) \quad R_n(f,x) = \sum_{k=1}^{n} f(x_{k,n})\, r_{k,n}(x)$$

and

$$(1.3.18) \quad \begin{aligned} \overline{R}_n(f,x) &= \sum_{k=1}^{n} f(x_{k,n})\, r_{k,n}(x) \\ &+ f'(1)\, \sigma_{1,n}(x) + f'(-1)\, \sigma_{2,n}(x) \end{aligned}$$

We refer to the paper of Szabodos and Varma [35] for explicit forms. The following are their convergence theorems.

<u>Theorem 1.13</u> (Szabodos and Varma). Let $f \in C[-1,1]$. Then

$$(1.3.19) \quad |f(x) - R_n(f,x)| = O\left(\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right)\right) \text{ for } -1 \le x \le 1$$

and $n = 1, 2, \ldots$

<u>Theorem 1.14</u> (Szabodos and Varma). Let f be a function such that $f' \in C[-1,1]$. Then

$$(1.3.20) \quad |f(x) - \overline{R}_n(f,x)| = O\left(\frac{\sqrt{1-x^2}}{n^2}\right) \sum_{k=1}^{n} \omega\left(f', \frac{1}{k}\right)$$

for $-1 \le x \le 1$ and $n = 1, 2, \ldots.$ A direct consequence of Theorem 1.12 is that (1.3.14) holds. This resolves Turán's Problem XXXI in a slightly different context.

For more results on Birkhoff interpolation, we refer the reader to the book of Lorentz [22].

Lacunary Spline Interpolation

In the 1970s, 1980s, and 1990s, several papers appeared in which (0,2), (0,3), and (0,1,3) interpolation problems were solved using polynomial splines and piecewise polynomials. We can classify many of these results into three groups.

Before proceeding, we define by $S_{n,q}^{(r)}$ the class of splines $S(x)$ such that

(1.4.1) i) $S(x) \in C^r[0,1]$

ii) $S(x)$ is a polynomial of degree q in $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$ where

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1.$$

In this first group, the data to be interpolated are prescribed at the joints of a spline as well as at the endpoints of the interval. A. Meir and A. Sharma [23] were the first ones to consider the case of (0,2) interpolation on equidistant knots. They have shown that for arbitrary lacunary data $\{y_i\}_{i=0}^{n}$ and $\{y_i''\}_{i=0}^{n}$, there exists a unique (up to the boundary conditions) quintic spline $S_n(x) \in C^3[0,1]$ with joints at $\frac{i}{n}$ $(i = 0, 1, \ldots, n)$ such that

$S_n(\frac{i}{n}) = y_i$, $S_n''(\frac{i}{n}) = y_i''$ (n odd). The boundary conditions are

$S_n'''(0) = y_0'''$ and $S_n'''(1) = y_n'''$. Moreover, if the given values

$\{y_i\}$, $\{y_i''\}$, and $\{y_o''', y_n'''\}$ are the values and the second and third derivative values, respectively, of a function f satisfying $f \in C^4[0,1]$, Meir and Sharma proved the following convergence theorem.

Theorem 1.15 (Meir and Sharma). For the unique quintic spline $S_n(x)$ that interpolates (0,2) data as discussed, we have

(1.4.2) $$\|S_n^{(r)} - f^{(r)}\| \le 75n^{r-3}\,\omega(f^{(4)}, \frac{1}{n}) + 8n^{r-4}\|f^{(4)}\|$$

for r = 0,1,2,3.

Subsequently, S. Demko pointed out that because of the ill-poised nature of the interpolant defined in the preceding, for a given function $f \in C^6[0,1]$, the error $\|f - S_n\|$ where S_n interpolates f (as described above), is not of optimal order as a function of mesh length [9]. He further gave justification to this claim. On the other hand, S. Demko also pointed out that the situation changes if, instead of considering (0,2) interpolation by splines, we consider the (0,3) case based on equidistant knots. Consider arbitrary lacunary data $\{y_i\}_{i=0}^n$, $\{y_i'''\}_{i=0}^n$, and $\{y_o'', y_n''\}$. There exists a unique quintic spline $S_n(x) \in C^3[0,1]$ with joints at $\frac{i}{n}$ $(i = 0, 1, \ldots, n)$ such that

$S_n(\frac{i}{n}) = y_i$, $S_n'''(\frac{i}{n}) = y_i'''$, $S_n''(0) = y_o''$, and $S_n''(1) = y_n''$. The system of equations that uniquely determines $S_n(x)$ turns out to be tridiagonal dominant, and consequently, the rate of convergence is of the same order as that of best approximation by quintic C^3 splines, provided the interpolated data corresponds to the function approximated.

$\left(y_i = f(x_i),\ y_i''' = f'''(x_i),\ y_o'' = f''(x_o),\ y_n'' = f''(x_n),\ f \in C^3[0,1]\right)$.

The second group of results deals with special piecewise polynomial methods for solving (0,2), (0,2,3), and (0,2,4) problems. We refer to the work of Fawzy [13] [14] [15]. Later, Fawzy and Schumaker [16] defined construction methods for solving the general lacunary interpolation problem.

On the positive side, these methods are shown to deliver the optimal order of approximation while being relatively easy to construct. One possible defect remarked by them is that their proposed methods produce only piecewise polynomials. We refer here to remark 1 on page 424[16]. Here, one should note that the data are prescribed at the knots only.

For the third group of results dealing with lacunary interpolation by splines, we refer to the papers of A.K. Varma [42] [43], J. Prasad and A.K. Varma [28], Gary

Howell and A.K. Varma [20]. Here, we allow certain data to be prescribed at the midpoints of the joints, in addition to at the joints of the spline.

Howell and Varma [20] obtained deficient quartic splines of the class $C^2[0,1]$ which interpolate lacunary data (function values at the midpoints of the joints, second derivative values at the joints, and function values at the endpoints of the interval).

They obtained the following convergence theorem for these splines.

<u>Theorem 1.16</u> (Howell and Varma). Let $f \in C^1[0,1]$. Then, for the unique quartic spline $S_n(x)$ associated with f and satisfying the above conditions, we have

(1.4.3) $|S_n^{(r)} - f^{(r)}(x)| \leq C_{r,l} h^{l-r} \omega(f^{(l)},h)$, r = 0,1 and l = 2,3,4, and

(1.4.4) $|S_n^{(r)}(x) - f^{(r)}(x)| \leq B_{r,5} h^{5-r} \max_{0 \leq x \leq 1} |f^{(5)}(x)|$, r = 0,1

and l = 5, where h is the mesh length.

The splines in this group are determined by tridiagonal dominant systems. In fact, we now present a case where the spline is actually determined by a diagonal matrix. Prasád and Varma obtained the first such case. They prescribed function values at the joints and midpoints of the joints, third derivative values at the midpoints of the joints, and

first derivative values at the endpoints of the interval. They proved the following convergence theorem.

Theorem 1.17 (Prasád and Varma). Let $f \in C^1[0,1]$. Then, for the unique quintic spline $S_n(x)$ associated with f and satisfying the above conditions, we have

$$(1.4.5) \qquad |S_n^{(r)}(x) - f^{(r)}(x)| \leq C_{r,l} h^{l-r} \omega(f^{(l)}, h), \quad r = 0,1,2$$

and $l = 3, 4, 5$, and

$$(1.4.6) \qquad |S_n^{(r)}(x) - f^{(r)}(x)| \leq B_{r,l}\, h^{6-r} \max_{0 \leq x \leq 1} |f^{(6)}(x)|,$$

$r = 0,1,2$ where h is the mesh length.

This concludes our discussion on lacunary spline interpolation.

Markov Type Inequalities

In 1889, A.A. Markov proved the following.

Theorem 1.18 (Markov). If $P_n(x)$ is a real algebraic polynomial of degree n such that $|P_n(x)| \leq 1$ on the interval $-1 \leq x \leq 1$, then we have

$$(1.5.1) \qquad \max_{-1 \leq x \leq 1} |P_n'(x)| \leq n^2.$$

Later, A. Zygmund [49] proved

Theorem 1.19 (Zygmund). If f is a trigonometric polynomial of order n and $p \ge 1$, then

$$(1.5.2) \qquad \left\{\frac{1}{2\pi}\int_0^{2\pi}|f'(\theta)|^p d\theta\right\}^{\frac{1}{p}} \le n\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|^p d\theta\right\}^{\frac{1}{p}}.$$

Hill, Szegö and Tamerkin [19] extended this type of inequality to algebraic polynomials on the interval [-1,1] in the form

$$(1.5.3) \qquad \left(\int_{-1}^{1}|P_n'(x)|^p dx\right)^{\frac{1}{p}} \le An^2\left(\int_{-1}^{1}|P_n(x)|^p dx\right)^{\frac{1}{p}}$$

where $p \ge 1$ and A is independent of n and $P_n(x)$. They noted that the problem of obtaining the best constant in (1.5.3) is extremely difficult. Later, B.D. Bojanov [7] proved the following extension of the Markov Inequality.

Theorem 1.20 (Bojanov). Let $1 \le p < \infty$. Then for every real algebraic polynomial of degree n, we have

$$(1.5.4) \qquad \left(\int_{-1}^{1}|P_n'(x)|^p dx\right)^{\frac{1}{p}} \le \left(\int_{-1}^{1}|T_n'(x)|^p dx\right)^{\frac{1}{p}} \max_{-1\le x\le 1}|P_n(x)|$$

where $T_n(x) = \cos n\theta$, $x = \cos\theta$.

The following problem was raised by P. Turán at a conference held in Varna, Bulgaria (1970). Let $\phi(x) \ge 0$ for $-1 \le x \le 1$ and consider the class $P_{n,\phi}$ of all polynomials

$P_n(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most n such that $|P_n(x)| \le \phi(x)$ for $-1 \le x \le 1$. How large can $\max_{-1 \le x \le 1} |P_n^{(k)}(x)|$ become if $P_n(x)$ is an arbitrary polynomial belonging to $P_{n,\phi}$? Important contributions to the problem of Turán have been made by Rahman and his associates. The results we state involve the circular $\left(\phi(x) = \sqrt{1-x^2}\right)$ and the parabolic majorants $(\phi(x) = 1 - x^2)$.

<u>Theorem 1.21</u> (Rahman). If $P_n(x)$ is an algebraic polynomial of degree n such that $|P_n(x)| \le \sqrt{1-x^2}$ for $-1 \le x \le 1$, then

$$\max_{-1 \le x \le 1} |P_n'(x)| \le 2\,(n-1)\,. \tag{1.5.5}$$

Equality if and only if

$$P_n(x) = (1 - x^2)\, u_{n-2}(x)\,,\quad u_{n-2}(x) = \frac{\sin(n-1)\theta}{\sin\theta}\,,\quad x = \cos\theta.$$

<u>Theorem 1.22</u> (Rahman and Watt). For given $n \ge 3$, let

$$\lambda_r = \lambda_{r,n} = \cos\left(\frac{r\pi}{n-2}\right),\quad r = 0,\ 1,\ \ldots,\ n-2. \tag{1.5.6}$$

If $P(x) = (1 - x^2)q(x)$ is a polynomial of degree at most n such that $|q(\lambda_r)| \le 1$ for $r = 0, 1, \ldots, n-2$, then

$$\|P^{(k)}\| \le |\tau_n^{(k)}(1)| \quad \text{for } k = 3,\ 4,\ \ldots \tag{1.5.7}$$

where $\tau_n(x) = (1-x^2)\,T_{n-2}(x)\,,\ T_{n-2}(x) = \cos\,(n-2)\theta,\ x = \cos\theta.$

Further, if P(x) is real for real x, then

$$(1.5.8) \qquad |P^{(k)}(x+iy)| \le |\tau_n^{(k)}(1+iy)| \text{ for } (x,y) \in [-1,1] \times \mathbb{R}$$

and k = 3, 4, ...

For other interesting results, we refer to the works of Rahman and associates [25] [26] [27] [29] [30] [31].

A natural extension of these ideas is to investigate similar problems in the L^p norm. We shall state two such results obtained by Varma and associates in the L^2 norm.

<u>Theorem 1.23</u> (Varma). Let $P_{n+1}(x)$ by any real algebraic polynomial of degree at most n + 1 such that $|P_{n+1}(x)| \le \sqrt{1-x^2}$ for $-1 \le x \le 1$, then

$$(1.5.9) \qquad \int_{-1}^{1} \left[P_{n+1}^{(j)}(x)\right]^2 (1-x^2)^{\frac{1}{2}}\, dx \le \int_{-1}^{1} \left[f_o^{(j)}(x)\right]^2 (1-x^2)^{\frac{1}{2}}\, dx$$

for j = 1, 2, 3 where $f_o(x) = (1-x^2)\, u_{n-1}(x)$, $u_{n-1}(x) = \dfrac{\sin n\theta}{\sin\theta}$, $x = \cos\theta$. Equality if and only if $P_{n+1}(x) = \pm f_o(x)$.

<u>Theorem 1.24</u> (Varma, Mills, and Smith). Let $P_{n+2}(x)$ be any real algebraic polynomial of degree at most n + 2 such that $|P_{n+2}(x)| \le 1-x^2$ for $-1 \le x \le 1$, then

$$(1.5.10) \qquad \int_{-1}^{1} \left[P_{n+2}''(x)\right]^2 dx \le \int_{-1}^{1} \left[f_1''(x)\right]^2 dx$$

where $f_1(x) = (1 - x^2)T_n(x)$, $T_n(x) = \cos n\theta$, $x = \cos\theta$. Equality if and only if $P_{n+2}(x) = \pm f_1(x)$.

For other results in the L^2 norm, we refer to the work of Varma and associates [44] [45] [46].

CHAPTER TWO
EXPLICIT REPRESENTATION OF A (0;2) PROCESS

Introduction and Main Results

Define

(2.1.1) $\quad -1 = t_{o,n} < t_{1,n} < \ldots, t_{n,n} = 1$

to be the zeros of $(1 - x^2)P_{n-1}(x)$, and

(2.1.2) $\quad -1 < x_{2,n} < x_{3,n}, \ldots < x_{n-1,n} < 1$

to be the zeros of $P_{n-1}'(x)$. Here, $P_{n-1}(x)$ denotes the Legendre polynomial of degree n - 1 with normalization

(2.1.3) $\quad P_{n-1}(1) = 1.$

The following theorem is a direct result of Lemma2 in a paper by Akhlaghi and Sharma [2].

Theorem 2A

Given arbitary values $(a_{o,n}, a_{1,n}, \ldots, a_{n,n})$ and $(b_{2,n}, b_{3,n}, \ldots, b_{n-1,n})$, there exists a unique real algebraic polynomial $R_n(x)$ of degree 2n - 2, such that

(2.1.4) $\quad R_n(t_{j,n}) = a_{j,n}$ for j = 1, 2, ..., n - 1,

(2.1.5) $\quad R_n(-1) = a_{o,n}$, $R_n(1) = a_{n,n}$,

(2.1.6) $\quad R_n''(x_{j,n}) = b_{j,n}$ for j = 2, 3, ..., n - 1.

We note that the above theorem places no restriction on n being even or odd. This is in contrast to other similar processes that have been studied.

We now present our results on explicit representation. Given arbitrary values $(a_{o,n}, a_{1,n}, \ldots, a_{n,n})$ and $(b_{2,n}, b_{3,n}, \ldots, b_{n-1,n})$, we wish to find the explicit form of the polynomial $R_n(x)$ of degree $\le 2n - 2$ such that (2.1.4) - (2.1.6) hold. For $R_n(x)$ we evidently have the form

$$R_n(x) = \sum_{k=0}^{n} a_{k,n}\, r_{k,n}(x) + \sum_{k=2}^{n-1} b_{k,n}\, \rho_{k,n}(x)\,, \tag{2.1.7}$$

where the polynomials $r_{k,n}(x)$ and $\rho_{k,n}(x)$ are the fundamental polynomials of the first and second kind. These polynomials are of degree $\le 2n - 2$ and are uniquely determined by the following conditions.

$$r_{k,n}(t_{j,n}) = \begin{cases} 0 & \text{for } j \ne k \\ 1 & \text{for } j = k \end{cases}, \quad r''_{k,n}(x_{i,n}) = r_{k,n}(\pm 1) = 0\,, \quad k = 1, 2, \ldots, n-1, \tag{2.1.8}$$

$$\begin{aligned} &r_{o,n}(t_{j,n}) = r_{n,n}(t_{j,n}) = 0\,, \quad r_{o,n}(-1) = r_{n,n}(1) = 1\,, \\ &r_{0,n}(1) = r_{n,n}(-1) = 0\,, \qquad r''_{o,n}(x_{i,n}) = r''_{n,n}(x_{i,n}) = 0 \end{aligned} \tag{2.1.9}$$

$$\rho_{k,n}(t_{j,n}) = \rho_{k,n}(\pm 1) = 0\,, \quad \rho''_{k,n}(x_{i,n}) = \begin{cases} 0 & \text{for } i \ne k \\ 1 & \text{for } i = k \end{cases}, \quad k = 2, 3, \ldots, n-1, \tag{2.1.10}$$

$(j = 1, 2, \ldots, n - 1$ and $i = 2, 3, \ldots, n - 1)$.

The following theorem presents the explicit representation of these fundamental polynomials.

Theorem 2.1

The fundamental polynomials $\rho_{k,n}(x)$ and $r_{k,n}(x)$ are given by

$$\rho_{k,n}(x) = -P_{n-1}(x)\left[\frac{(1-x_{k,n}^2)}{n(n-1)\;P_{n-1}^3(x_{k,n})}\right] \tag{2.1.11}$$

$$\cdot\sum_{r=2}^{n-1}\frac{(2r-1)\,P_{r-1}'(x_{k,n})\,\pi_r(x)}{r(r-1)\,[n(n-1)+r(r-1)]},$$

$k = 2, 3, \ldots, n-1$,

$$r_{k,n}(x) = A_{k,n}(x) - B_{k,n}(x),\quad k = 1, 2, \ldots, n-1, \tag{2.1.12}$$

where

$$A_{k,n}(x) = \frac{(1-x^2)\;P_{n-1}(x)\;P_{n-1}'(x)}{(1-t_{k,n}^2)\;(x-t_{k,n})\;\left[P_{n-1}'(t_{k,n})\right]^2}$$

and

$$B_{k,n}(x) = \frac{2n(n-1)\,P_{n-1}(x)}{(1-t_{k,n}^2)^2\left[P_{n-1}'(t_{k,n})\right]^3}$$

$$\cdot\sum_{r=2}^{n-1}\frac{(2r-1)\;P_{r-1}(t_{k,n})\;\pi_r(x)}{n(n-1)+r(r-1)},$$

$$r_{n,n}(x) = r_{o,n}(-x) = \frac{(1+x)}{2}\,P_{n-1}(x)\;P_{n-2}(x)$$

$$+\,P_{n-1}(x)\sum_{r=2}^{n-2}\frac{(2r-1)\;\pi_r(x)}{n(n-1)+r(r-1)} \tag{2.1.13}$$

$$+\,\frac{(n-2)}{2(n-1)^2}\,P_{n-1}(x)\;\pi_{n-1}(x)$$

The relative simplicity of these explicit forms is crucial to proving the convergence results in the next chapter.

Preliminaries

Here, we list various known results used in the proofs of the next section. The following identities were taken from Chapter Three of Sansone [32].

(2.2.1) $$(1-x^2)\, P''_{n-1}(x) - 2x\, P'_{n-1}(x) + n(n-1)\, P_{n-1}(x) = 0$$

(2.2.2) $$(1-x^2)\, P'''_{n-1}(x) - 4xP''_{n-1}(x) + (n-2)(n+1)\, P'_{n-1}(x) = 0$$

(2.2.3) $$\pi'_r(x) = -r(r-1)\, P_{r-1}(x)$$

(2.2.4) $$(1-x^2)\, P'_{r-1}(x) = (r-1)\, P_{r-2}(x) - (r-1)\, xP_{r-1}(x)$$

(2.2.5) $$xP'_{r-1}(x) - P'_{r-2}(x) = (r-1)\, P_{r-1}(x)$$

(2.2.6) $$P'_{r-1}(x) - xP'_{r-2}(x) = (r-1)\, P_{r-2}(x)$$

(2.2.7) $$P'_{r-1}(x) - P'_{r-3}(x) = (2r-3)\, P_{r-2}(x)$$

(2.2.8) $$(r-2)\, \frac{P_{r-3}(x) - P_{r-2}(x)}{1-x} = P'_{r-2}(x) + P'_{r-3}(x)$$

(2.2.9) $$\sum_{s=1}^{r} (2s-1)\, P_{s-1}(x)\, P_{s-1}(y) = r\, \frac{P_{r-1}(x)\, P_r(y) - P_r(x)\, P_{r-1}(y)}{y-x}$$

The above identity is known as the Christoffel formula of summation. Differentiating (2.2.9) once gives the following.

(2.2.10) $$\sum_{s=2}^{r} (2s-1)\, P'_{s-1}(x)\, P_{s-1}(y) = r\left[\frac{P'_{r-1}(x)\, P_r(y) - P'_r(x)\, P_{r-1}(y)}{y-x} + \frac{P_{r-1}(x)\, P_r(y) - P_r(x)\, P_{r-1}(y)}{(y-x)^2}\right]$$

From Szegö [38], we write

$$(2.2.11)\qquad r\sum_{s=2}^{r}\frac{(2s-1)\,P_{s-1}'(x)\,P_{s-1}'(y)}{s(s-1)}=\frac{P_r'(x)\,P_{r-1}'(y)-P_{r-1}'(x)\,P_r'(y)}{x-y}.$$

Proof of Theorem 2.1

We first prove (2.1.11). In view of the uniqueness theorem, it is enough to verify that $\rho_{k,n}(x)$, as stated in (2.1.11), indeed satisfies the conditions in (2.1.10). The first condition clearly holds.

Fix any $k \in \{2, 3, \ldots, n-1\}$. From (2.2.1) and (2.2.3),

$$(2.3.1)\qquad \begin{aligned}&\frac{d^2}{dx^2}\left[P_{n-1}(x)\ \pi_r(x)\right]_{x=x_{j,n}}=P_{n-1}(x_{j,n})\ \pi_r''(x_{j,n})\\&+P_{n-1}''(x_{j,n})\ \pi_r(x_{j,n})=-r(r-1)\ P_{n-1}(x_{j,n})\ P_{r-1}'(x_{j,n})\\&-n(n-1)\ P_{n-1}(x_{j,n})\ P_{r-1}(x_{j,n})\,.\end{aligned}$$

Hence,

$$(2.3.2)\qquad \begin{aligned}&\rho_{k,n}''(x_{j,n})\\&=\frac{(1-x_{k,n}^2)\ P_{n-1}(x_{j,n})}{n(n-1)\ P_{n-1}^3(x_{k,n})}\\&\cdot\sum_{r=2}^{n-1}\frac{(2r-1)\ P_{r-1}'(x_{k,n})\ P_{r-1}'(x_{j,n})}{r(r-1)}\,.\end{aligned}$$

From (2.2.11), we now have

$$\rho''_{k,n}(x_{j,n})$$

$$(2.3.3) \qquad = \frac{(1-x_{k,n}^2)\ P''_{n-1}(x_{k,n})\ P'_{n-2}(x_{k,n})\ P_{n-1}(x_{j,n})}{n(n-1)^2\ P_{n-1}^3(x_{k,n})}$$

$$\cdot\left[\frac{P'_{n-1}(x_{j,n})}{(x_{j,n}-x_{k,n})\ P''_{n-1}(x_{k,n})}\right].$$

From (2.2.1) and (2.2.5),

$$(2.3.4) \qquad \rho''_{k,n}(x_{j,n}) = \frac{P_{n-1}(x_{j,n})}{P_{n-1}(x_{k,n})}\left[\frac{P'_{n-1}(x_{j,n})}{(x_{j,n}-x_{k,n})\ P''_{n-1}(x_{k,n})}\right]$$

Recalling the fundamental functions of Lagrange interpolation, we see that

$$(2.3.5) \qquad \rho''_{k,n}(x_{j,n}) = 0 \text{ for } j \neq k \text{ and}$$

$$(2.3.6) \qquad \rho''_{k,n}(x_{k,n}) = 1.$$

We conclude that all conditions in (2.1.10) hold.

Next, we turn our attention to (2.1.12). Again, we need only verify the conditions in (2.1.8) hold, and all but the second derivative conditions are clear from (2.1.12).

Fix any $k \in \{1, 2, \ldots, n-1\}$. From (2.1.12), we have

$$(2.3.7) \qquad A_{k,n}(x) = \frac{\pi_n(x)\ l_{k,n}(x)}{(1-t_{k,n}^2)\ P'_{n-1}(t_{k,n})} \text{ where}$$

$$l_{k,n}(x) = \frac{P_{n-1}(x)}{(x-t_{k,n})\ P'_{n-1}(t_{k,n})}.$$

Next, we differentiate twice to get

$$(2.3.8)\qquad \begin{aligned} A''_{k,n}(x_{j,n}) &= \frac{-2n(n-1)\ P_{n-1}(x_{j,n})\ l'_{k,n}(x_{j,n})}{(1-t^2_{k,n})\ P'_{n-1}(t_{k,n})} \\ &= \frac{2n(n-1)\ P^2_{n-1}(x_{j,n})}{(1-t^2_{k,n})\ \left[P'_{n-1}(t_{k,n})\right]^2\ (x_{j,n}-t_{k,n})^2}. \end{aligned}$$

Now, from (2.3.1), we have

$$(2.3.9)\qquad \begin{aligned} &B''_{k,n}(x_{j,n}) \\ &= \frac{-2n(n-1)\ P_{n-1}(x_{j,n})}{(1-t^2_{k,n})^2\left[P'_{n-1}(t_{k,n})\right]^3} \\ &\cdot \sum_{r=2}^{n-1} (2r-1)\ P_{r-1}(t_{k,n})\ P'_{r-1}(x_{j,n}). \end{aligned}$$

We first apply (2.2.10) and secondly (2.2.4) to (2.3.9) and obtain

$$(2.3.10)\qquad \begin{aligned} B''_{k,n}(x_{j,n}) &= \frac{2n(n-1)^2\ P^2_{n-1}(x_{j,n})\ P_{n-2}(t_{k,n})}{(1-t^2_{k,n})^2\left[P'_{n-1}(t_{k,n})\right]^3\ (x_{j,n}-t_{k,n})^2} \\ &= \frac{2n(n-1)\ P^2_{n-1}(x_{j,n})}{(1-t^2_{k,n})\ \left[P'_{n-1}(t_{k,n})\right]^2\ (x_{j,n}-t_{k,n})^2}. \end{aligned}$$

Hence,

$$(2.3.11)\qquad r''_{k,n}(x_{j,n}) = A''_{k,n}(x_{j,n}) - B''_{k,n}(x_{j,n}) = 0$$

for $j = 2, 3, \ldots, n-1$.

We conclude that all conditions in (2.1.8) hold.

Lastly, we prove (2.1.13). We begin by observing that $r_{0,n}(x) = r_{n,n}(-x)$ follows from uniqueness and symmetry of the zeros of $P_{n-1}(x)$ and $P'_{n-1}(x)$. We, therefore, need only

verify conditions (2.1.9). Specifically, we show that $r''_{n,n}(x_{j,n}) = 0$ for $j = 2, 3, \ldots, n-1$.

From (2.3.1), (2.2.10), and finally (2.2.8),

$$\frac{d^2}{dx^2}\left[\sum_{r=2}^{n-2} \frac{(2r-1)\ P_{n-1}(x)\,\pi_r(x)}{n(n-1)+r(r-1)}\right]_{x=x_{j,n}}$$

$$= -P_{n-1}(x_{j,n}) \sum_{r=2}^{n-2} (2r-1)\ P'_{r-1}(x_{j,n})\ P_{r-1}(1)$$

$$\begin{aligned} &= -(n-2)\ P_{n-1}(x_{j,n})\left[\frac{P'_{n-3}(x_{j,n}) - P'_{n-2}(x_{j,n})}{1-x_{j,n}}\right. \\ &\quad \left. + \frac{P_{n-3}(x_{j,n}) - P_{n-2}(x_{j,n})}{(1-x_{j,n})^2}\right] \end{aligned} \tag{2.3.12}$$

$$= \frac{P_{n-1}(x_{j,n})}{(1-x_{j,n})}\left[(n-3)\ P'_{n-2}(x_{j,n}) - (n-1)\ P'_{n-3}(x_{j,n})\right]$$

We now apply (2.3.1) and (2.3.10) to (2.1.13) and obtain

$$\begin{aligned} &r''_{n,n}(x_{j,n}) \\ &= \frac{(1+x_{j,n})}{2}\left[P''_{n-1}(x_{j,n})\,P_{n-2}(x_{j,n})\right. \\ &\quad \left. + P_{n-1}(x_{j,n})\ P''_{n-2}(x_{j,n})\right] + P_{n-1}(x_{j,n})\,P'_{n-2}(x_{j,n}) \\ &\quad + \frac{P_{n-1}(x_{j,n})}{1-x_{j,n}}\left[(n-3)\ P'_{n-2}(x_{j,n}) - (n-1)\ P'_{n-3}(x_{j,n})\right] \\ &\quad - (n-2)\ P_{n-1}(x_{j,n})\ P'_{n-2}(x_{j,n}) \end{aligned} \tag{2.3.13}$$

Hence,

$$\frac{(1-x_{j,n})}{P_{n-1}(x_{j,n})}\, r''_{n,n}(x_{j,n})$$

(2.3.14)
$$= \frac{(1-x_{j,n}^2)}{2P_{n-1}(x_{j,n})}\left[P''_{n-1}(x_{j,n})\, P_{n-2}(x_{j,n}) + P_{n-1}(x_{j,n})\, P''_{n-2}(x_{j,n})\right]$$

$$+ (1-x_{j,n})\, P'_{n-2}(x_{j,n}) + (n-3)\, P'_{n-2}(x_{j,n}) - (n-1)\, P'_{n-3}(x_{j,n})$$

$$- (n-2)\, (1-x_{j,n})\, P'_{n-2}(x_{j,n}).$$

From (2.2.1),

(2.3.15)
$$(1-x_{j,n}^2)\left[P''_{n-1}(x_{j,n})\, P_{n-2}(x_{j,n}) + P_{n-1}(x_{j,n})\, P''_{n-2}(x_{j,n})\right]$$

$$= 2P_{n-1}(x_{j,n})\left[x_{j,n}\, P'_{n-2}(x_{j,n}) - (n-1)^2\, P_{n-2}(x_{j,n})\right].$$

We now apply (2.3.15) to (2.3.14) and obtain

$$\frac{(1-x_{j,n})}{P_{n-1}(x_{j,n})}\, r''_{n,n}(x_{j,n})$$

(2.3.16)
$$= (n-2)\, x_{j,n}\, P'_{n-2}(x_{j,n}) - (n-1)^2\, P_{n-2}(x_{j,n})$$

$$- (n-1)\, P'_{n-3}(x_{j,n}).$$

From (2.2.6) and (2.2.7),

(2.3.17)
$$\frac{(1-x_{j,n})}{P_{n-1}(x_{j,n})}\, r''_{n,n}(x_{j,n}) = 0$$

Hence,

(2.3.18) $r''_{n,n}(x_{j,n}) = 0$ for j = 2, 3, ..., n - 1.

This concludes the proof of (2.1.13) and, consequently, the proof of Theorem 2.1.

CHAPTER THREE
CONVERGENCE RESULTS FOR A (0;2) PROCESS

Introduction and Main Results

Let f be a real valued function defined on the interval [-1,1]. We now define the linear operator $R_n(f;x)$ by

$$R_n(f;x) = \sum_{k=0}^{n} f(t_{k,n})\, r_{k,n}(x) \tag{3.1.1}$$

where

$$\begin{aligned} r_{k,n}(x) &= \frac{(1-x^2)\, P_{n-1}(x)\, P'_{n-1}(x)}{(1-t_{k,n})^2 (x-t_{k,n}) \left[P'_{n-1}(t_{k,n})\right]^2} \\ &- \frac{2n(n-1)\, P_{n-1}(x)}{(1-t^2_{k,n})^2 \left[P'_{n-1}(t_{k,n})\right]^3} \sum_{r=2}^{n-1} \frac{(2r-1)\, P_{r-1}(t_{k,n})\, \pi_r(x)}{n(n-1)+r(r-1)} \end{aligned} \tag{3.1.2}$$

for k = 1, 2, ..., n - 1

and

$$\begin{aligned} r_{n,n}(x) = r_{o,n}(-x) &= \frac{(1+x)}{2} P_{n-1}(x) P_{n-2}(x) \\ &+ P_{n-1}(x) \sum_{r=2}^{n-2} \frac{(2r-1)\, \pi_r(x)}{n(n-1)+r(r-1)} \\ &+ \frac{(n-2)}{2(n-1)^2} P_{n-1}(x)\, \pi_{n-1}(x)\,. \end{aligned} \tag{3.1.3}$$

The formulas (3.1.2) and (3.1.3) are taken from Theorem 2.1 and the $t_{k,n}$ are defined as in (2.1.1).

We now present the main objective of this chapter.

Theorem 3.1

Let f be a continuous function on the interval [-1,1]. Then

$$(3.1.4) \qquad |R_n(f;x) - f(x)| \le C\,\omega\left(f; \frac{\sqrt{1-x^2}}{n}\right)$$

where C is a constant independent of f, x, and n, and $\omega(f;\delta)$ represents the usual modulus of continuity.

We remark that Theorem 3.1 is a discrete interpolatory example of the Teljakovskii Theorem (Theorem 1.5).

Preliminaries

This section is comprised of a listing of known results which are necessary in the subsequent sections. All constants, stated or implied, are independent of x, k, and n. Also, we define $x = \cos\theta$ and $t_{k,n} = \cos\theta_{k,n}$ where $t_{k,n}$'s are as defined in (2.1.1).

We begin with the formula for summation by parts.

$$(3.2.1) \qquad \sum_{k=M}^{m} a_k b_k = \sum_{k=m}^{M-1} A_k\,(b_k - b_{k+1}) + A_M b_M \text{ where } A_k = \sum_{r=m}^{k} a_r$$

Recalling the definition of the $t_{k,n}$'s, we have [38]

$$(3.2.2) \qquad (1 - t_{k,n}^2) > d_1 \frac{k^2}{(n-1)^2},\ k = 1, 2, \ldots, \left[\frac{n-1}{2}\right],$$

$$(3.2.3) \qquad (1 - t_{k,n}^2) > d_2 \frac{(n-k)^2}{(n-1)^2},\ k = \left[\frac{n-1}{2}\right] + 1, \ldots, n-1,$$

(3.2.4) $\quad |P'_{n-1}(t_{k,n})| \geq \dfrac{d_3(n-1)^2}{k^{3/2}}$, k = 1, 2, ..., $\left[\dfrac{n-1}{2}\right]$,

(3.2.5) $\quad |P'_{n-1}(t_{k,n})| \geq \dfrac{d_4(n-1)^2}{(n-k)^{3/2}}$, k = $\left[\dfrac{n-1}{2}\right]$ + 1, ..., n - 1,

(3.2.6) $\quad |\theta_{k,n} - \theta_{k+1,n}| > \dfrac{d_5}{n}$.

Note (3.2.6) is a direct result of Bruns' Inequality. The following inequalities may be found in Szegö [38].

(3.2.7) $\quad |P_n(x)| = O\left(\dfrac{1}{\sqrt{n}\sin\theta}\right)$ for -1 < x < 1

(3.2.8) $\quad |P_n(x)| = O(1)$ for $-1 \leq x \leq 1$

(3.2.9) $\quad |P'_n(x)| = O\left(\sqrt{\dfrac{n}{\sin^3\theta}}\right)$ for -1 < x < 1

(3.2.10) $\quad |P'_n(x)| = O\left(\dfrac{n}{\sin\theta}\right)$ for -1 < x < 1

From (3.2.2) - (3.2.5),

(3.2.11) $\quad \dfrac{1}{(1-t_{k,n}^2)^2\,|P'_{n-1}(t_{k,n})|^3} = O\left(\dfrac{\sqrt{\sin\theta_{k,n}}}{n^{3/2}}\right)$, k = 1, 2, ...,

n - 1.

From (3.2.9) - (3.2.11),

$$(3.2.12)\qquad \frac{1}{(1-t_{k,n}^2)^2\,|P_{n-1}'(t_{k,n})|^3} = \frac{(1-t_{k,n}^2)^2\,|P_{n-1}'(t_{k,n})|^3}{(1-t_{k,n}^2)^4\,|P_{n-1}'(t_{k,n})|^6} = O\left(\frac{\sin^{3/2}\theta_{k,n}}{\sqrt{n}}\right),\ k = 1, 2, \ldots, n-1.$$

The subsequent two results can be found in a paper by Szabodos and Varma [36].

$$(3.2.13)\qquad |P_n(x) + P_{n+1}(x)| = O\left(\sqrt{\frac{\sin\theta}{n}}\right) \text{ for } -1 \le x \le \frac{1}{2}$$

$$(3.2.14)\qquad |P_n'(x) + P_{n+1}'(x)| = O\left(\sqrt{\frac{n}{\sin\theta}}\right) \text{ for } -1 < x \le \frac{1}{2}$$

From (2.2.8) and (3.2.8),

$$(3.2.15)\qquad |P_n'(x) + P_{n+1}'(x)| = O(n) \text{ for } -1 \le x \le \frac{1}{2}.$$

From (2.2.4) and (3.2.9), we may write

$$(3.2.16)\qquad |P_{n-2}(t_{k,n})| = O\left(\sqrt{\frac{\sin\theta_{k,n}}{n}}\right),\ k = 1, 2, \ldots, n-1.$$

Finally, from (3.2.2) and (3.2.3),

$$(3.2.17)\qquad \sum_{|\theta-\theta_{k,n}|\ge\frac{c}{n}} \frac{1}{n^2\sin^2\frac{|\theta-\theta_{k,n}|}{2}} = O(1)$$

and

(3.2.18) $$\sum_{|\theta-\theta_{k,n}|\geq \frac{c}{n}} \frac{1}{n^3 \sin^3 \frac{|\theta - \theta_{k,n}|}{2}} = O(1).$$

From Erdos [12],

(3.2.19) $$\left| \frac{P_{n-1}(x)}{(x - t_{k,n})\ P'_{n-1}(t_{k,n})} \right| = O(1)$$

Lemmas

Lemma 3.1

For $-1 < x \leq \frac{1}{2}$ and $-1 < y \leq \frac{1}{2}$, we have

(3.3.1) $$r\ |P_{r-1}(x)\ P_r(y) - P_r(x)\ P_{r-1}(y)| = O\left(\frac{\sin\theta + \sin\phi}{\sqrt{\sin\theta \sin\phi}} \right)$$

and

(3.3.2) $$|P_{r-1}(x)\, P_r(y) - P_r(x)\, P_{r-1}(y)| = O\left(\frac{\sqrt{\sin\theta} + \sqrt{\sin\phi}}{\sqrt{r}} \right)$$

where $x = \cos\theta$ and $y = \cos\phi$.

Proof:

$$r\,|P_{r-1}(x)\,P_r(y) - P_r(x)\,P_{r-1}(y)|$$

$$= r\,|[P_{r-1}(x)\,P_r(y) + P_r(x)\,P_r(y)] - [P_r(x)\,P_r(y) + P_r(x)\,P_{r-1}(y)]|$$

$$\leq r\,|P_{r-1}(x) + P_r(x)|\,|P_r(y)| + r\,|P_r(y) + P_{r-1}(y)|\,|P_r(x)|$$

$$= O\left(\sqrt{\frac{\sin\theta}{\sin\phi}} + \sqrt{\frac{\sin\phi}{\sin\theta}}\right) = O\left(\frac{\sin\theta + \sin\phi}{\sqrt{\sin\theta\,\sin\phi}}\right)$$

In the above, we used estimates (3.2.7) and (3.2.13). To prove (3.3.2), we use estimates (3.2.8) and (3.2.13).

$$|P_{r-1}(x)\,P_r(y) - P_r(x)\,P_{r-1}(y)| \leq |P_{r-1}(x) + P_r(x)|\,|P_r(y)|$$

$$+ |P_r(y) + P_{r-1}(y)|\,|P_r(x)| = O\left(\sqrt{\frac{\sin\theta}{r}} + \sqrt{\frac{\sin\phi}{r}}\right)$$

Lemma 3.2

For $-1 < x \leq \frac{1}{2}$ and $-1 < y \leq \frac{1}{2}$, we have

(3.3.3) $$|P'_{r-1}(x)\,P_r(y) - P'_r(x)\,P_{r-1}(y)| = O\left(\frac{\sin\theta + \sin\phi}{\sqrt{\sin^3\theta\,\sin\phi}}\right)$$

and

(3.3.4) $$|P'_{r-1}(x)\,P_r(y) - P'_r(x)\,P_{r-1}(y)| = O\left(\frac{\sqrt{r}(\sin\theta + \sin\phi)}{\sin\theta\,\sqrt{\sin\phi}}\right)$$

where $x = \cos\theta$ and $y = \cos\phi$.

Proof:

$$|P'_{r-1}(x)\,P_r(y) - P'_r(x)\,P_{r-1}(y)| = \Big|\big[P'_{r-1}(x)\,P_r(y) + P'_r(x)\,P_r(y)\big]$$

$$- \big[P'_r(x)\,P_r(y) + P'_r(x)\,P_{r-1}(y)\big]\Big| \le |P'_{r-1}(x) + P'_r(x)|\;|P_r(y)|$$

$$+ |P_r(y) + P_{r-1}(y)|\;|P'_r(x)| = O\left(\frac{1}{\sqrt{\sin\theta\,\sin\phi}} + \sqrt{\frac{\sin\phi}{\sin^3\theta}}\right)$$

$$= O\left(\frac{\sin\theta + \sin\phi}{\sqrt{\sin^3\theta\,\sin\phi}}\right)$$

Here, we used the estimates (3.2.7), (3.2.9), (3.2.13), and (3.2.14). To prove (3.3.4), we use (3.2.7), (3.2.10), (3.2.13), and (3.2.15).

$$|P'_{r-1}(x)\,P_r(y) - P'_r(x)\,P_{r-1}(y)| \le |P'_{r-1}(x) + P'_r(x)|\;|P_r(y)|$$

$$+ |P_r(y) + P_{r-1}(y)|\;|P'_r(x)| = O\left(\sqrt{\frac{r}{\sin\phi}} + \frac{\sqrt{r\sin\phi}}{\sin\theta}\right)$$

$$= O\left(\frac{\sqrt{r}\,(\sin\theta + \sin\phi)}{\sin\theta\sqrt{\sin\phi}}\right).$$

Lemma 3.3

$$\sum_{s=2}^{r}\big[P'_{s-1}(x)\,P_s(t_{k,n}) - P'_s(x)\,P_{s-1}(t_{k,n})\big]$$

$$= \left[\frac{r(P_{r-1}(x) - P_r(x)}{1-x}\,P_r(t_{k,n}) + 1\right]$$

$$- P'_r(x)\,P_r(t_{k,n}) - r\,\frac{P_{r-1}(x)\,P_r(t_{k,n}) - P_r(x)\,P_{r-1}(t_{k,n})}{t_{k,n} - x}$$

Proof:

$$\sum_{s=2}^{r}\left[P_{s-1}'(x)\,P_s(t_{k,n}) - P_s'(x)\,P_{s-1}(t_{k,n})\right]$$

$$= \sum_{s=2}^{r} P_{s-1}'(x)\,P_s(t_{k,n}) - \sum_{s=2}^{r} P_{s-2}'(x)\,P_{s-1}(t_{k,n})$$

$$- \sum_{s=2}^{r}(2s-1)\,P_{s-1}(x)\,P_{s-1}(t_{k,n})$$

$$= \sum_{s=2}^{r} P_{s-1}'(x)\,P_s(t_{k,n}) - \sum_{s=2}^{r-1} P_{s-1}'(x)\,P_s(t_{k,n})$$

$$- \sum_{s=2}^{r}(2s-1)\,P_{s-1}(x)\,P_{s-1}(t_{k,n})$$

$$= P_{r-1}'(x)\,P_r(t_{k,n}) - \left[r\,\frac{P_{r-1}(x)\,P_r(t_{k,n}) - P_r(x)\,P_{r-1}(t_{k,n})}{t_{k,n}-x} - 1\right]$$

The above equalities used (2.2.7) and (2.2.9). We next apply (2.2.8) to obtain

$$\sum_{s=2}^{r}\left[P_{s-1}'(x)\,P_s(t_{k,n}) - P_s'(x)\,P_{s-1}(t_{k,n})\right]$$

$$= \left[-P_r'(x) + r\,\frac{P_{r-1}(x) - P_r(x)}{1-x}\right]P_r(t_{k,n})$$

$$- \left[r\,\frac{P_{r-1}(x)\,P_r(t_{k,n}) - P_r(x)\,P_{r-1}(t_{k,n})}{t_{k,n}-x} - 1\right].$$

The lemma follows.

Lemma 3.4

$$\sum_{s=2}^{r} 2s^2 \left[P_{s-1}(x)\, P_s(t_{k,n}) - P_s(x)\, P_{s-1}(t_{k,n})\right]$$

$$= (1 - x^2)(r + 1)\left[\frac{P_r'(x)\, P_{r+1}(t_{k,n}) - P_{r+1}'(x)\, P_r(t_{k,n})}{t_{k,n} - x}\right.$$

$$\left. + \frac{P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})}{(t_{k,n} - x)^2}\right] \tag{3.3.5}$$

$$- (1 - t_{k,n}^2)(r + 1)\left[\frac{P_r'(t_{k,n})\, P_{r+1}(x) - P_{r+1}'(t_{k,n})\, P_r(x)}{x - t_{k,n}}\right.$$

$$\left. + \frac{P_r(t_{k,n})\, P_{r+1}(x) - P_{r+1}(t_{k,n})\, P_r(x)}{(x - t_{k,n})^2}\right]$$

$$- (r + 1)\, r\left[P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})\right] + 2(x - t_{k,n})$$

Proof: From (2.2.4), we may write

$$\sum_{s=2}^{r} (2s + 1)\, s\left[P_{s-1}(x)\, P_s(t_{k,n}) - P_s(x)\, P_{s-1}(t_{k,n})\right]$$

$$= \sum_{s=2}^{r} (2s + 1)\, P_s(t_{k,n})\left[(1 - x^2)\, P_s'(x) + sxP_s(x)\right]$$

$$- \sum_{s=2}^{r} (2s + 1)\, P_s(x)\left[(1 - t_{k,n}^2)\, P_s'(t_{k,n}) + s\, t_{k,n} P_s(t_{k,n})\right] \tag{3.3.6}$$

$$= (1 - x^2) \sum_{s=2}^{r} (2s + 1)\, P_s(t_{k,n})\, P_s'(x)$$

$$- (1 - t_{k,n}^2) \sum_{s=2}^{r} (2s + 1)\, P_s(x)\, P_s'(t_{k,n})$$

$$+ (x - t_{k,n}) \sum_{s=2}^{r} (2s + 1)\; s\, P_s(x)\, P_s(t_{k,n})$$

We now work with the last sum in (3.3.6). From summation by parts (3.2.1),

$$
\begin{aligned}
&\sum_{s=2}^{r} (2s+1)\, s\, P_s(x)\, P_s(t_{k,n}) \\
&= \sum_{s=2}^{r-1} [s - (s+1)] \sum_{m=2}^{s} (2m+1)\, P_m(x)\, P_m(t_{k,n}) \\
&+ r \sum_{m=2}^{r} (2m+1)\, P_m(x)\, P_m(t_{k,n}) \\
&= - \sum_{s=2}^{r-1} (s+1) \left[\frac{P_s(x)\, P_{s+1}(t_{k,n}) - P_{s+1}(x)\, P_s(t_{k,n})}{t_{k,n} - x} \right] \\
&+ \sum_{s=2}^{r-1} (1 + 3xt_{k,n}) + r(r+1) \\
&\cdot \left[\frac{P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})}{t_{k,n} - x} \right] \\
&- r(1 + 3xt_{k,n})\,.
\end{aligned}
\tag{3.3.7}
$$

We used (2.2.9) in the second equality of (3.3.7). Now,

$$
\begin{aligned}
&(x - t_{k,n}) \sum_{s=2}^{r} (2s+1)\, s\, P_s(x)\, P_s(t_{k,n}) \\
&= \sum_{s=2}^{r} s\, [P_{s-1}(x)\, P_s(t_{k,n}) - P_s(x)\, P_{s-1}(t_{k,n})] \\
&- 2\, [P_1(x)\, P_2(t_{k,n}) - P_2(x)\, P_1(t_{k,n})] \\
&+ (x - t_{k,n})(r-2)(1 + 3xt_{k,n}) \\
&- r(r+1)\, [P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})] \\
&- (x - t_{k,n})\, r\, (1 + 3xt_{k,n}) \\
&= \sum_{s=2}^{r} s\, [P_{s-1}(x)\, P_s(t_{k,n}) - P_s(x)\, P_{s-1}(t_{k,n})] \\
&- (x - t_{k,n})(= 3xt_{k,n}) \\
&- r(r+1)\, [P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})].
\end{aligned}
\tag{3.3.8}
$$

Together (3.3.6) and (3.3.8) imply that

$$\sum_{s=2}^{r} (2s+1)\, s\left[P_{s-1}(x)\,P_s(t_{k,n}) - P_s(x)\,P_{s-1}(t_{k,n})\right]$$

$$- \sum_{s=2}^{r} s\left[P_{s-1}(x)\,P_s(t_{k,n}) - P_s(x)\,P_{s-1}(t_{k,n})\right]$$

(3.3.9) $$= (1-x^2) \sum_{s=2}^{r} (2s+1)\, P_s(t_{k,n})\, P_s'(x)$$

$$- (1-t_{k,n}^2) \sum_{s=2}^{r} (2s+1)\, P_s(x)\, P_s'(t_{k,n})$$

$$- (x - t_{k,n})(1 + 3x t_{k,n})$$

$$- r(r+1)\left[P_r(x)\,P_{r+1}(t_{k,n}) - P_{r+1}(x)\,P_r(t_{k,n})\right]$$

With further calculations on (3.3.9), we obtain

$$\sum_{s=2}^{r} 2s^2\left[P_{s-1}(x)\,P_s(t_{k,n}) - P_s(x)\,P_{s-1}(t_{k,n})\right]$$

$$= (1-x^2) \sum_{s=2}^{r+1} (2s-1)\, P_{s-1}(t_{k,n})\, P_{s-1}'(x)$$

(3.3.10) $$- (1-t_{k,n}^2) \sum_{s=2}^{r+1} (2s-1)\, P_{s-1}(x)\, P_{s-1}'(t_{k,n})$$

$$- 3(1-x^2)\, t_{k,n} + 3(1-t_{k,n}^2)\, x$$

$$- (x - t_{k,n})(1 + 3x t_{k,n})$$

$$- r(r+1)\left[P_r(x)\,P_{r+1}(t_{k,n}) - P_{r+1}(x)\,P_r(t_{k,n})\right].$$

From (2.2.10),

$$
\begin{aligned}
&\sum_{s=2}^{r} 2s^2 \left[P_{s-1}(x) P_s(t_{k,n}) - P_s(x) P_{s-1}(t_{k,n})\right] \\
&= (1 - x^2)(r + 1) \left[\frac{P_r'(x) P_{r+1}(t_{k,n}) - P_{r+1}'(x) P_r(t_{k,n})}{t_{k,n} - x} \right. \\
&\quad \left. + \frac{P_r(x) P_{r+1}(t_{k,n}) - P_{r+1}(x) P_r(t_{k,n})}{(t_{k,n} - x)^2} \right] \\
&\quad - (1 - t_{k,n}^2)(r + 1) \left[\frac{P_r'(t_{k,n}) P_{r+1}(x) - P_{r+1}'(t_{k,n}) P_r(x)}{x - t_{k,n}} \right. \\
&\quad \left. + \frac{P_r(t_{k,n}) P_{r+1}(x) - P_{r+1}(t_{k,n}) P_r(x)}{(x - t_{k,n})^2} \right] + 2(x - t_{k,n}) \\
&\quad - r(r + 1) \left[P_r(x) P_{r+1}(t_{k,n}) - P_{r+1}(x) P_r(t_{k,n})\right]
\end{aligned}
\tag{3.3.11}
$$

The proof of the lemma is finished.

<u>Lemma 3.5</u>

For $-1 < x \leq 0$,

$$
\begin{aligned}
&\frac{1}{|t_{k,n} - x|} \sum_{r=2}^{n-1} \left(\frac{1}{r^2 n^4} + \frac{1}{n^5} \right) \left| \sum_{s=2}^{r} 2s^2 [P_{s-1}(x) P_s(t_{k,n}) \right. \\
&\quad \left. - P_s(x) P_{s-1}(t_{k,n})] \right| \\
&= O\left(\frac{1}{n\sqrt{\sin\theta \sin\theta_{k,n}}} \left[\frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}} \right. \right. \\
&\quad \left. \left. + \frac{1}{n^3 \sin^3 \frac{|\theta - \theta_{k,n}|}{2}} + \frac{1}{n} \right] \right) \text{ for } i = 3,\ 4.
\end{aligned}
\tag{3.3.12}
$$

<u>Proof</u>: From Lemma 3.4,

$$\left| \sum_{s=2}^{r} 2s^2 \left[P_{s-1}(x)\, P_s(t_{k,n}) - P_s(x)\, P_{s-1}(t_{k,n})\right] \right|$$

$$(3.3.13)\qquad \begin{aligned} &\le \frac{(1-x^2)(r+1)}{|t_{k,n}-x|} \left|P_r'(x)\, P_{r+1}(t_{k,n}) - P_{r+1}'(x)\, P_r(t_{k,n})\right| \\ &+ \frac{(1-x^2)(r+1)}{|t_{k,n}-x|^2} \left|P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})\right| \\ &+ \frac{(1-t_{k,n}^2)(r+1)}{|t_{k,n}-x|} \left|P_r'(t_{k,n})\, P_{r+1}(x) - P_{r+1}'(t_{k,n})\, P_r(x)\right| \\ &+ \frac{(1-t_{k,n}^2)(r+1)}{|t_{k,n}-x|^2} \left|P_r(t_{k,n})\, P_{r+1}(x) - P_{r+1}(t_{k,n})\, P_r(x)\right| \\ &+ r(r+1)\left|P_r(x)\, P_{r+1}(t_{k,n}) - P_{r+1}(x)\, P_r(t_{k,n})\right| + 2|x - t_{k,n}| \end{aligned}$$

We now break into two cases.

<u>Case 1</u>: $-1 \le t_{k,n} \le \frac{1}{2}$

To get (3.3.12), we need to analyze the order of the following terms. From Lemma 3.2 (3.3.3),

$$(3.3.14)\qquad \begin{aligned} &\sum_{r=2}^{n-i} \left(\frac{1}{r^2 n^4} + \frac{1}{n^5}\right) \frac{(1-x^2)(r+1)}{|t_{k,n}-x|^2} \\ &\cdot \left|P_r'(x)\, P_{r+1}(t_{k,n}) - P_{r+1}'(x)\, P_r(t_{k,n})\right| \\ &= O\left(\sum_{r=2}^{n-i} \left(\frac{1}{r n^4}\right) \frac{\sin^2\theta\,(r+1)(\sin\theta + \sin\theta_{k,n})}{|t_{k,n}-x|^2 \sqrt{\sin^3\theta \sin\theta_{k,n}}}\right) \\ &= O\left(\frac{1}{n^3 \sin^2\frac{|\theta-\theta_{k,n}|}{2} \sqrt{\sin\theta \sin\theta_{k,n}}}\right) \end{aligned}$$

and

$$\sum_{r=2}^{n-i}\left(\frac{1}{r^2n^4}+\frac{1}{n^5}\right)\frac{(1-t_{k,n}^2)(r+1)}{|t_{k,n}-x|^2}$$

$$\cdot\left|P_r'(t_{k,n})P_{r+1}(x)-P_{r+1}'(t_{k,n})P_r(x)\right|$$

(3.3.15)

$$=O\left(\sum_{r=2}^{n-i}\left(\frac{1}{rn^4}\right)\frac{\sin^2\theta_{k,n}(r+1)(\sin\theta+\sin\theta_{k,n})}{|t_{k,n}-x|^2\sqrt{\sin^3\theta_{k,n}\sin\theta}}\right)$$

$$=O\left(\frac{1}{n^3\sin^2\frac{|\theta-\theta_{k,n}|}{2}\sqrt{\sin\theta\sin\theta_{k,n}}}\right).$$

From Lemma 3.1 (3.3.1),

$$\sum_{r=2}^{n-i}\left(\frac{1}{r^2n^4}+\frac{1}{n^5}\right)\frac{(1-x^2)(r+1)}{|t_{k,n}-x|^3}$$

$$\cdot\left|P_r(x)P_{r+1}(t_{k,n})-P_{r+1}(x)P_r(t_{k,n})\right|$$

(3.3.16)

$$=O\left(\frac{1}{n^4}\sum_{r=2}^{n-i}\left(\frac{1}{r^2}+\frac{1}{n}\right)\frac{\sin^2\theta(r+1)(\sin\theta+\sin\theta_{k,n})}{|t_{k,n}-x|^3\,r\sqrt{\sin\theta\sin\theta_{k,n}}}\right)$$

$$=O\left(\frac{1}{n^4\sin^3\frac{|\theta-\theta_{k,n}|}{2}\sqrt{\sin\theta\sin\theta_{k,n}}}\right),$$

$$\sum_{r=2}^{n-i}\left(\frac{1}{r^2n^4}+\frac{1}{n^5}\right)\frac{(1-t_{k,n}^2)(r+1)}{|t_{k,n}-x|^3}$$

$$\cdot\left|P_r(t_{k,n})P_{r+1}(x)-P_{r+1}(t_{k,n})P_r(x)\right|$$

(3.3.17)

$$=O\left(\frac{1}{n^4}\sum_{r=2}^{n-i}\left(\frac{1}{r^2}+\frac{1}{n}\right)\frac{\sin^2\theta_{k,n}(r+1)(\sin\theta+\sin\theta_{k,n})}{|t_{k,n}-x|^3\,r\sqrt{\sin\theta\sin\theta_{k,n}}}\right)$$

$$=O\left(\frac{1}{n^4\sin^3\frac{|\theta-\theta_{k,n}|}{2}\sqrt{\sin\theta\sin\theta_{k,n}}}\right),$$

and

$$\sum_{r=2}^{n-i}\left(\frac{1}{r^2n^4}+\frac{1}{n^5}\right)\frac{r(r+1)}{|t_{k,n}-x|}$$

$$\cdot|P_r(x)P_{r+1}(t_{k,n})-P_{r+1}(x)P_r(t_{k,n})|$$

(3.3.18)

$$=O\left(\sum_{r=2}^{n-i}\left(\frac{1}{rn^4}\right)\frac{r(r+1)(\sin\theta+\sin\theta_{k,n})}{|t_{k,n}-x|\ r\sqrt{\sin\theta\sin\theta_{k,n}}}\right)$$

$$=O\left(\frac{1}{n^3\sin^2\frac{|\theta-\theta_{k,n}|}{2}\sqrt{\sin\theta\sin\theta_{k,n}}}\right).$$

Also,

(3.3.19)

$$\sum_{r=2}^{n-i}\left(\frac{1}{r^2n^4}+\frac{1}{n^5}\right)(2)=O\left(\frac{1}{n^2}\right)$$

$$=O\left(\frac{1}{n^2\sqrt{\sin\theta\sin\theta_{k,n}}}\right).$$

In case 1, the lemma follows.

<u>Case 2</u>: $\frac{1}{2}\le t_{k,n}<1$

Since $-1<x\le 0$, we note that $|t_{k,n}-x|\ge\frac{1}{2}$. An elementary application of estimates (3.2.8) and (3.2.10) along with some calculations on the various terms in (3.3.14) - (3.3.19) gives

$$\frac{1}{|t_{k,n} - x|} \sum_{r=2}^{n-i} \left(\frac{1}{r^2 n^4} + \frac{1}{n^5} \right)$$

$$(3.3.20) \qquad \cdot \left| \sum_{s=2}^{r} 2s \left[P_{s-1}(x) P_x(t_{k,n}) - P_s(x) P_{s-1}(t_{k,n}) \right] \right|$$

$$= O\left(\frac{1}{n^2} \right) = O\left(\frac{1}{n^2 \sqrt{\sin\theta \sin\theta_{k,n}}} \right).$$

The lemma follows.

<u>Lemma 3.6</u>

For $-1 < x \le 0$,

$$\frac{1}{n^4 |t_{k,n} - x|} \left| \sum_{s=2}^{n-i} 2s^2 \left[P_{s-1}(x) P_s(t_{k,n}) - P_s(x) P_{s-1}(t_{k,n}) \right] \right|$$

$$(3.3.21) \qquad = O\left(\frac{1}{n \sqrt{\sin\theta \sin\theta_{k,n}}} \left[\frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}} \right. \right.$$

$$\left. \left. + \frac{1}{n^3 \sin^3 \frac{|\theta - \theta_{k,n}|}{2}} + \frac{1}{n} \right] \right) \text{ for } i = 2, 3.$$

Proof: We begin by utilizing Lemma 3.4 as in (3.3.13).

$$\begin{aligned}
&\left|\sum_{s=2}^{n-i} 2s^2 [P_{s-1}(x) P_s(t_{k,n}) - P_s(x) P_{s-1}(t_{k,n})]\right| \\
&\le \frac{(1-x^2)(n-i+1)}{|t_{k,n}-x|} |P'_{n-i}(x) P_{n-i+1}(t_{k,n}) \\
&- P'_{n-i+1}(x) P_{n-i}(t_{k,n})| + \frac{(1-x^2)(n-i+1)}{|t_{k,n}-x|^2} \\
&\cdot |P_{n-i}(x) P_{n-i+1}(t_{k,n}) - P_{n-i+1}(x) P_{n-i}(t_{k,n})| \\
&+ \frac{(1-t_{k,n}^2)(n-i+1)}{|t_{k,n}-x|} |P'_{n-i}(t_{k,n}) P_{n-i+1}(x) \\
&- P'_{n-i+1}(t_{k,n}) P_{n-i}(x)| + \frac{(1-t_{k,n}^2)(n-i+1)}{|t_{k,n}-x|^2} \\
&\cdot |P_{n-i}(t_{k,n}) P_{n-i+1}(x) - P_{n-i+1}(t_{k,n}) P_{n-i}(x)| \\
&+ (n-i)(n-i+1)|P_{n-i}(x) P_{n-i+1}(t_{k,n}) \\
&- P_{n-i+1}(x) P_{n-i}(t_{k,n})| + 2|x-t_{k,n}|
\end{aligned} \tag{3.3.22}$$

We break into two cases.

Case 1: $-1 < t_{k,n} \le \frac{1}{2}$

As in the previous lemma, we need to estimate six different terms. From Lemma 3.2 (3.3.3),

$$\begin{aligned}
&\frac{(1-x^2)(n-i+1)}{n^4|t_{k,n}-x|^2} |P'_{n-i}(x) P_{n-i+1}(t_{k,n}) \\
&- P'_{n-i+1}(x) P_{n-i}(t_{k,n})| \\
&= O\left(\frac{1}{n^3 \sin^2 \frac{|\theta-\theta_{k,n}|}{2} \sqrt{\sin\theta \sin\theta_{k,n}}}\right)
\end{aligned} \tag{3.3.23}$$

and

$$\frac{(1 - t_{k,n}^2)(n - i + 1)}{n^4 |t_{k,n} - x|^2} |P'_{n-i}(t_{k,n}) P_{n-i+1}(x)$$

(3.3.24) $$- P'_{n-i+1}(t_{k,n}) P_{n-i}(x)|$$

$$= O\left(\frac{1}{n^3 \sin^2 \frac{|\theta - \theta_{k,n}|}{2} \sqrt{\sin\theta \sin\theta_{k,n}}}\right).$$

From Lemma 3.1 (3.3.1),

$$\frac{(1 - x^2)(n - i + 1)}{n^4 |t_{k,n} - x|^3} |P_{n-i}(x) P_{n-i+1}(t_{k,n})$$

(3.3.25) $$- P_{n-i+1}(x) P_{n-i}(t_{k,n})|$$

$$= O\left(\frac{1}{n^4 \sin^3 \frac{|\theta - \theta_{k,n}|}{2} \sqrt{\sin\theta \sin\theta_{k,n}}}\right),$$

$$\frac{(1 - t_{k,n}^2)(n - i + 1)}{n^4 |t_{k,n} - x|^3} |P_{n-i}(t_{k,n}) P_{n-i+1}(x)$$

(3.3.26) $$- P_{n-i+1}(t_{k,n}) P_{n-i}(x)|$$

$$= O\left(\frac{1}{n^4 \sin^3 \frac{|\theta - \theta_{k,n}|}{2} \sqrt{\sin\theta \sin\theta_{k,n}}}\right),$$

and

$$\frac{(n-i)(n-i+1)}{n^4|t_{k,n}-x|}|P_{n-i}(x)P_{n-i+1}(t_{k,n})$$

(3.3.27) $$- P_{n-i+1}(x)P_{n-i}(t_{k,n})|$$

$$= O\left(\frac{1}{n^3 \sin^2\frac{|\theta-\theta_{k,n}|}{2}\sqrt{\sin\theta\sin\theta_{k,n}}}\right).$$

Also,

(3.3.28) $$\frac{2}{n^4} \le \left(\frac{C}{n^2\sqrt{\sin\theta\sin\theta_{k,n}}}\right).$$

The lemma in case 1 follows.

<u>Case 2</u>: $\frac{1}{2} \le t_{k,n} < 1$.

Here we note that $|t_{k,n} - x| \ge \frac{1}{2}$ and apply (3.2.8) and (3.2.10) to the various terms in (3.3.23) - (3.3.28). The result is

(3.3.29) $$\frac{1}{n^4|t_{k,n}-x|}\left|\sum_{s=2}^{n-i} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right|$$

$$= O\left(\frac{1}{n^2\sqrt{\sin\theta\sin\theta_{k,n}}}\right)$$

and the proof of the lemma is complete.

Lemma 3.7

For $-1 < x \le 0$,

$$|x - t_{k,n}|\,|r_{k,n}(x)| = O\left(\frac{\sin\theta}{n}\left[\frac{1}{n^2\sin^2\frac{|\theta - \theta_{k,n}|}{2}}\right.\right.$$

(3.3.30)

$$\left.\left.+ \frac{1}{n^3\sin^3\frac{|\theta - \theta_{k,n}|}{2}} + \frac{1}{n}\right]\right) \text{ for } k = 1, 2, \ldots, n-1.$$

Proof: To begin with, define

(3.3.31) $\alpha_{r,n} = n(n - 1) + r(r - 1)$ *and* $\lambda_{r,n} = \dfrac{1}{\alpha_{r,n}}$.

From summation by parts (3.2.1) and (2.2.10),

$$\sum_{r=2}^{n-1} \frac{(2r - 1)\,P_{r-1}(t_{k,n})\,P'_{r-1}(x)}{n(n - 1) + r(r - 1)}$$

$$= \sum_{r=2}^{n-2} [\lambda_{r,n} - \lambda_{r+1,n}] \sum_{s=2}^{r} (2s - 1)\,P_{s-1}(t_{k,n})\,P'_{s-1}(x)$$

$$+ \lambda_{n-1,n} \sum_{s=2}^{n-1} (2s - 1)\,P_{s-1}(t_{k,n})\,P'_{s-1}(x)$$

(3.3.32)

$$= 2\sum_{r=2}^{n-2} \frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}}\left[\frac{P'_{r-1}(x)\,P_r(t_{k,n}) - P'_r(x)\,P_{r-1}(t_{k,n})}{t_{k,n} - x}\right]$$

$$+ 2\sum_{r=2}^{n-2} \frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}}\left[\frac{P_{r-1}(x)\,P_r(t_{k,n}) - P_r(x)\,P_{r-1}(t_{k,n})}{(t_{k,n} - x)^2}\right]$$

$$- \frac{(n - 1)}{2(n - 1)^2}\left[\frac{P'_{n-1}(x)\,P_{n-2}(t_{k,n})}{t_{k,n} - x} + \frac{P_{n-1}(x)\,P_{n-2}(t_{k,n})}{(t_{k,n} - x)^2}\right]$$

Recalling the definition of $r_{k,n}(x)$ given in (2.1.12), we may write

(3.3.33) $(x - t_{k,n})\,r_{k,n}(x) = Z_1 + Z_2 + Z_3 + Z_4 + Z_5$

where

$$Z_1 = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3} \tag{3.3.34}$$
$$\cdot \sum_{r=2}^{n-2} \frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}} \left[P_{r-1}'(x)P_r(t_{k,n}) - P_r'(x)P_{r-1}(t_{k,n})\right],$$

$$Z_2 = \frac{-4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3} \tag{3.3.35}$$
$$\cdot \sum_{r=2}^{n-2} \frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}} \left[\frac{P_{r-1}(x)P_r(t_{k,n}) - P_r(x)P_{r-1}(t_{k,n})}{x - t_{k,n}}\right],$$

$$Z_3 = \frac{(1-x^2)P_{n-1}(x)P_{n-1}'(x)}{(1-t_{k,n}^2)\ [P_{n-1}'(t_{k,n})]^2}, \tag{3.3.36}$$

$$Z_4 = \frac{-n(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3}\left[P_{n-1}'(x)P_{n-2}(t_{k,n})\right], \tag{3.3.37}$$

and

$$Z_5 = \frac{n(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3}\left[\frac{P_{n-1}(x)P_{n-2}(t_{k,n})}{x - t_{k,n}}\right]. \tag{3.3.38}$$

We begin by working with Z_2. From (3.2.1), estimates (3.2.7) and (3.2.11), and Lemmas 3.5 and 3.6,

$$|Z_2| = O\Bigg(\frac{\sin^{3/2}\theta\sqrt{\sin\theta_{k,n}}}{|t_{k,n} - x|}\Bigg[\sum_{r=2}^{n-3}\left|\frac{1}{\alpha_{r,n}\alpha_{r+1,n}} - \frac{1}{\alpha_{r+1,n}\alpha_{r+2,n}}\right|$$

$$\cdot\left|\sum_{s=2}^{r} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right| + \frac{1}{\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\cdot\left|\sum_{s=2}^{n-2} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right|\Bigg]\Bigg)$$

(3.3.39)
$$= O\Bigg(\frac{\sin^{3/2}\theta\sqrt{\sin\theta_{k,n}}}{|t_{k,n} - x|}\Bigg[\sum_{r=2}^{n-3}\left(\frac{1}{r^2n^4} + \frac{1}{n^5}\right)$$

$$\cdot\left|\sum_{s=2}^{r} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right|$$

$$+ \frac{1}{n^4}\left|\sum_{s=2}^{n-2} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right|\Bigg]\Bigg)$$

$$= O\left(\frac{\sin\theta}{n}\left[\frac{1}{n^2\sin^2\frac{|\theta - \theta_{k,n}|}{2}} + \frac{1}{n^3\sin^3\frac{|\theta - \theta_{k,n}|}{2}} + \frac{1}{n}\right]\right).$$

We now work with Z_1.

From (3.2.1) and Lemma 3.3,

$$Z_1 = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}$$

$$\cdot\Bigg\{\sum_{r=2}^{n-3}\left[\frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}} - \frac{(r+1)^2}{\alpha_{r+1,n}\alpha_{r+2,n}}\right]$$

(3.3.40)
$$\cdot\sum_{s=2}^{r}\left[P'_{s-1}(x)P_s(t_{k,n}) - P'_s(x)P_{s-1}(t_{k,n})\right]$$

$$+ \frac{(n-2)^2}{\alpha_{n-2,n}\alpha_{n-1,n}}\sum_{s=2}^{n-2}\left[P'_{s-1}(x)P_s(t_{k,n}) - P'_s(x)P_{s-1}(t_{k,n})\right]\Bigg\}$$

$$= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6$$

where

$$Q_1 = \frac{-4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3}$$

(3.3.41) $$\cdot \sum_{r=2}^{n-3} \frac{(2r+1)[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}$$

$$\cdot \left[\frac{r(P_{r-1}(x) - P_r(x))}{1-x} P_r(t_{k,n}) + 1 \right],$$

$$Q_2 = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3}$$

(3.3.42) $$\cdot \sum_{r=2}^{n-3} \frac{(2r+1)[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}$$

$$\cdot \left[P_r'(x) P_r(t_{k,n}) \right],$$

$$Q_3 = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3}$$

(3.3.43) $$\cdot \sum_{r=2}^{n-3} \frac{(2r+1)\, r\, [n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}(t_{k,n}-x)}$$

$$\cdot [P_{r-1}(x) P_r(t_{k,n}) - P_r(x) P_{r-1}(t_{k,n})],$$

$$Q_4 = \frac{4n(n-1)(1-x^2)P_{n-1}(x)(n-2)^2}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3\,\alpha_{n-2,n}\alpha_{n-1,n}}$$

(3.3.44)

$$\cdot \left[\frac{(n-2)(P_{n-3}(x) - P_{n-2}(x))}{1-x} P_{n-2}(t_{k,n}) + 1 \right],$$

$$Q_5 = \frac{-4n(n-1)(1-x^2)P_{n-1}(x)(n-2)^2}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3\,\alpha_{n-2,n}\alpha_{n-1,n}} \tag{3.3.45}$$
$$\cdot\left[P_{n-2}'(x)P_{n-2}(t_{k,n})\right],$$

and

$$Q_6 = \frac{-4n(n-1)(1-x^2)P_{n-1}(x)(n-2)^3}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3\,\alpha_{n-2,n}\alpha_{n-1,n}} \tag{3.3.46}$$
$$\cdot\left[\frac{P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n})}{t_{k,n}-x}\right].$$

We now estimate Q_1. From (3.2.7) and (3.2.11),

$$|Q_1| = O\Bigg(\sin^{3/2}\theta\sqrt{\sin\theta_{k,n}} \tag{3.3.47}$$
$$\cdot\sum_{r=2}^{n-3}\frac{1}{n^3\sqrt{\sin\theta\sin\theta_{k,n}}}\Bigg) = O\left(\frac{\sin\theta}{n^2}\right).$$

We next skip to Q_3. From summation by parts (3.2.1),

$$\sum_{r=2}^{n-3} \frac{2(2r+1)r[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}$$

$$\cdot [P_{r-1}(x)P_r(t_{k,n}) - P_r(x)P_{r-1}(t_{k,n})]$$

$$= \sum_{r=2}^{n-4}\left[\frac{(2r+1)[n(n-1)-r(r+1)]}{r\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}\right.$$

$$\left. - \frac{(2r+3)[n(n-1)-(r+1)(r+2)]}{(r+1)\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}}\right] \tag{3.3.48}$$

$$\cdot \sum_{s=2}^{r} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]$$

$$+ \frac{(2n-5)[n(n-1)-(n-3)(n-2)]}{(n-3)\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\cdot \sum_{s=2}^{n-3} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})].$$

Together, (3.3.43) and (3.3.48) imply that

$$|\mathfrak{D}_3| = O\left(\frac{n^2(1-x^2)|P_{n-1}(x)|}{(1-t_{k,n}^2)^2|P'_{n-1}(t_{k,n})|^3}\right.$$

$$\cdot\left[\sum_{r=2}^{n-4}\left(\frac{1}{r^2n^4}+\frac{1}{n^5}\right)\left|\sum_{s=2}^{r} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right|\right. \tag{3.3.49}$$

$$\left.\left. + \frac{1}{n^5}\left|\sum_{s=2}^{n-3} 2s^2[P_{s-1}(x)P_s(t_{k,n}) - P_s(x)P_{s-1}(t_{k,n})]\right|\right]\right).$$

From (3.2.7), (3.2.11), Lemmas 3.5 and 3.6, along with (3.3.49),

$$
(3.3.50)\qquad
\begin{aligned}
|Q_3| = O\Biggl(\frac{\sin\theta}{n}\Biggl[&\frac{1}{n^2\sin^2\frac{|\theta-\theta_{k,n}|}{2}}\\
&+\frac{1}{n^3\sin^3\frac{|\theta-\theta_{k,n}|}{2}}+\frac{1}{n}\Biggr]\Biggr).
\end{aligned}
$$

Using (2.2.4), we may rewrite Z_3 as

$$
(3.3.51)\qquad Z_3 = \frac{(n-2)(1-x^2)P_{n-2}(t_{k,n})P_{n-1}(x)P'_{n-1}(x)}{(1-t_{k,n}^2)^2\,[P'_{n-1}(t_{k,n})]^3}.
$$

We now look at the order of the following combination of terms.

$$
(3.3.52)\qquad
\begin{aligned}
Z_3+Z_4+Q_5 &= \frac{(1-x^2)P_{n-2}(t_{k,n})P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}\\
&\quad\cdot\left[(n-1)P'_{n-1}(x)-nP'_{n-1}(x)-\frac{4n(n-1)(n-2)^2P'_{n-2}(x)}{\alpha_{n-2,n}\alpha_{n-1,n}}\right]\\
&= \frac{-(1-x^2)P_{n-2}(t_{k,n})P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}\\
&\quad\cdot\left[\bigl(P'_{n-1}(x)+P'_{n-2}(x)\bigr)+\frac{(-2n+3)P'_{n-2}(x)}{(n-1)(n^2-3n+3)}\right].
\end{aligned}
$$

From (3.2.7), (3.2.14), and (3.2.11),

$$\frac{(1 - x^2)\,|P_{n-2}(t_{k,n})P_{n-1}(x)|}{(1 - t_{k,n}^2)^2\,|P_{n-1}'(t_{k,n})|^3} \tag{3.3.53}$$

$$\cdot\,|P_{n-1}'(x) + P_{n-2}'(x)| = O\left(\frac{\sin\theta}{n^2}\right).$$

From (3.2.8), (3.2.10), and (3.2.11)

$$\frac{(1 - x^2)\,|P_{n-2}(t_{k,n})P_{n-1}(x)|\,(2n - 3)P_{n-2}'(x)}{(1 - t_{k,n}^2)^2\,|P_{n-1}'(t_{k,n})|^3\,(n - 1)(n^2 - 3n + 3)} = O\left(\frac{\sin\theta}{n^2}\right). \tag{3.3.54}$$

Hence,

$$|Z_3 + Z_4 + Q_5| = O\left(\frac{\sin\theta}{n^2}\right). \tag{3.3.55}$$

We next look at another combination of terms.

$$Z_5 + Q_6 = \frac{n(1 - x^2)P_{n-1}(x)}{(x - t_{k,n})(1 - t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3}$$

$$\cdot\left[P_{n-1}(x)P_{n-2}(t_{k,n}) + \frac{4(n - 1)(n - 2)^3}{\alpha_{n-2,n}\alpha_{n-1,n}}\right.$$

$$\left.\cdot\,(P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n}))\right]$$

$$= \frac{n(1 - x^2)P_{n-1}(x)}{(x - t_{k,n})(1 - t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3} \tag{3.3.56}$$

$$\cdot\left[P_{n-1}(x)P_{n-2}(t_{k,n}) + P_{n-3}(x)P_{n-2}(t_{k,n})\right.$$

$$- P_{n-2}(x)P_{n-3}(t_{k,n}) + \frac{(-2n^2 + 6n - 5)}{(n - 1)(n^2 - 3n + 3)}$$

$$\left.\cdot\,(P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n}))\right].$$

From (3.2.7), (3.2.11), and Lemma 3.1 (3.3.1), we have

$$\frac{n(1-x^2)\,|P_{n-1}(x)|\,|-2n^2+6n-5|}{|x-t_{k,n}|(1-t_{k,n}^2)^2|P'_{n-1}(t_{k,n})|^3(n-1)(n^2-3n+3)}$$

(3.3.57)
$$\cdot\,|P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n})|$$
$$= O\left(\frac{\sin^2\theta\sqrt{\sin\theta_{k,n}}(\sin\theta+\sin\theta_{k,n})}{|x-t_{k,n}|n^3\sin\theta\sqrt{\sin\theta_{k,n}}}\right)$$
$$= O\left(\frac{\sin\theta}{n^3\sin\frac{|\theta-\theta_{k,n}|}{2}}\right) \text{ for } -1 < t_{k,n} \le \frac{1}{2}.$$

From (3.2.7), (3.2.8), and (3.2.11),

$$\frac{n(1-x^2)\,|P_{n-1}(x)|\,|-2n^2+6n-5|}{|x-t_{k,n}|\,(1-t_{k,n}^2)^2|P'_{n-1}(t_{k,n})|^3(n-1)(n^2-3n+3)}$$

(3.3.58)
$$\cdot\,|P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n})|$$
$$= O\left(\frac{\sin^{3/2}\theta\sqrt{\sin\theta_{k,n}}}{n^2}\right) = O\left(\frac{\sin\theta}{n^2}\right) \text{ for } \frac{1}{2} \le t_{k,n} < 1.$$

From the following identity [32],

(3.3.59) $$(n-2)P_{n-3}(x) = -(n-1)P_{n-1}(x) + (2n-3)xP_{n-2}(x),$$

we may write

$$P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n})$$

(3.3.60)
$$= \frac{(2n-3)}{(n-2)}(x-t_{k,n})P_{n-2}(t_{k,n})P_{n-2}(x)$$
$$- \frac{(n-1)}{(n-2)}P_{n-2}(t_{k,n})P_{n-1}(x).$$

Hence,

$$\frac{n(1 - x^2) P_{n-1}(x)}{(x - t_{k,n})(1 - t_{k,n}^2)^2 [P_{n-1}'(t_{k,n})]^3}$$

$$\cdot [P_{n-1}(x) P_{n-2}(t_{k,n}) + P_{n-3}(x) P_{n-2}(t_{k,n})$$

$$- P_{n-2}(x) P_{n-3}(t_{k,n})]$$

(3.3.61)

$$= \frac{n(1 - x^2) P_{n-1}(x)}{(x - t_{k,n})(1 - t_{k,n}^2)^2 [P_{n-1}'(t_{k,n})]^3}$$

$$\cdot \left[\frac{-1}{(n - 2)} P_{n-1}(x) P_{n-2}(t_{k,n}) + \frac{(2n - 3)}{(n - 2)} \right.$$

$$\left. \cdot (x - t_{k,n}) P_{n-2}(t_{k,n}) P_{n-2}(x) \right].$$

From (3.2.7), (3.2.11), and (3.2.16),

$$\frac{n(1 - x^2) P_{n-1}^2(x) \, |P_{n-2}(t_{k,n})|}{|x - t_{k,n}| (1 - t_{k,n}^2)^2 |P_{n-1}'(t_{k,n})|^3 (n - 2)}$$

(3.3.62) $$= O\left(\frac{\sin^2\theta \sin\theta_{k,n}}{n^3 \sin\theta \, |x - t_{k,n}|} \right)$$

$$= O\left(\frac{\sin\theta}{n^3 \sin\frac{|\theta - \theta_{k,n}|}{2}} \right).$$

From (3.2.7) and (3.2.11),

$$\frac{n(1-x^2)\,|P_{n-1}(x)\,P_{n-2}(t_{k,n})\,P_{n-2}(x)\,|\,(2n-3)}{(1-t_{k,n}^2)^2\,|P_{n-1}'(t_{k,n})\,|^3\,(n-2)} \tag{3.3.63}$$
$$= O\left(\frac{\sin\theta}{n^2}\right).$$

We conclude that

$$|Z_5 + Q_6| = O\left(\frac{\sin\theta}{n}\left[\frac{1}{n^2\sin^2\frac{|\theta-\theta_{k,n}|}{2}} + \frac{1}{n}\right]\right). \tag{3.3.64}$$

To estimate Q_4, we use (3.2.7) and (3.2.11) to obtain

$$|Q_4| = O\left(\frac{\sin\theta}{n^2}\right). \tag{3.3.65}$$

We now observe that we need only estimate Q_2. From summation by parts (3.2.1) and (2.2.10),

$$\sum_{r=2}^{n-3} \frac{(2r+1)[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}} P_r'(x) P_r(t_{k,n})$$

$$= \sum_{r=2}^{n-4} \left[\frac{n(n-1)-r(r+1)}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}} - \frac{n(n-1)-(r+1)(r+2)}{\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}} \right]$$

$$\cdot \sum_{s=2}^{r} (2s+1) P_s'(x) P_s(t_{k,n}) + \frac{n(n-1)-(n-3)(n-2)}{\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\cdot \sum_{s=2}^{n-3} (2s+1) P_s'(x) P_s(t_{k,n})$$

$$= \sum_{r=2}^{n-4} \frac{4(r+1)[2n(n-1)-r(r+2)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}}$$

(3.3.66)
$$\cdot \left\{ (r+1) \frac{P_r'(x) P_{r+1}(t_{k,n}) - P_{r+1}'(x) P_r(t_{k,n})}{t_{k,n} - x} \right.$$

$$\left. + (r+1) \frac{P_r(x) P_{r+1}(t_{k,n}) - P_{r+1}(x) P_r(t_{k,n})}{(t_{k,n} - x)^2} - 3t_{k,n} \right\}$$

$$+ \frac{2(2n-3)}{\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\cdot \left\{ (n-2) \frac{P_{n-3}'(x) P_{n-2}(t_{k,n}) - P_{n-2}'(x) P_{n-3}(t_{k,n})}{t_{k,n} - x} \right.$$

$$\left. + (n-2) \frac{P_{n-3}(x) P_{n-2}(t_{k,n}) - P_{n-2}(x) P_{n-3}(t_{k,n})}{(t_{k,n} - x)^2} - 3t_{k,n} \right\}.$$

Now,

(3.3.67) $Q_2 = R_1 + R_2 + R_3 + R_4 + R_5 + R_6$

where

$$R_1 = \frac{16n(n-1)(1-x^2) P_{n-1}(x)}{(1-t_{k,n}^2)^2 [P_{n-1}'(t_{k,n})]^3}$$

(3.3.68)
$$\cdot \sum_{r=2}^{n-4} \frac{(r+1)^2 [2n(n-1) - r(r+2)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}}$$

$$\cdot \left[\frac{P_r'(x) P_{r+1}(t_{k,n}) - P_{r+1}'(x) P_r(t_{k,n})}{t_{k,n} - x} \right],$$

$$R_2 = \frac{16n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}$$

$$(3.3.69) \qquad \cdot \sum_{r=2}^{n-4} \frac{(r+1)^2[2n(n-1)-r(r+2)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}}$$

$$\cdot \left[\frac{P_r(x)P_{r+1}(t_{k,n}) - P_{r+1}(x)P_r(t_{k,n})}{(t_{k,n}-x)^2} \right],$$

$$R_3 = \frac{-48n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}$$

$$(3.3.70) \qquad \cdot \sum_{r=2}^{n-4} \frac{(r+1)[2n(n-1)-r(r+2)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}} \; t_{k,n},$$

$$R_4 = \frac{8n(n-1)(n-2)(2n-3)(1-x^2)P_{n-1}(x)}{\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}$$

$$(3.3.71) \qquad \cdot \left[\frac{P'_{n-3}(x)P_{n-2}(t_{k,n}) - P'_{n-2}(x)P_{n-3}(t_{k,n})}{t_{k,n}-x} \right],$$

$$R_5 = \frac{8n(n-1)(n-2)(2n-3)(1-x^2)P_{n-1}(x)}{\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}$$

$$(3.3.72) \qquad \cdot \left[\frac{P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n})}{(t_{k,n}-x)^2} \right],$$

and

$$(3.3.73) \qquad R_6 = \frac{-24n(n-1)(2n-3)(1-x^2)P_{n-1}(x)\,t_{k,n}}{\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}.$$

We proceed to estimate R_1 - R_6. From (3.2.8), (3.2.11), and Lemma 3.1 (3.3.2),

(3.3.74)
$$|R_2| = O\Bigg(\sqrt{n\sin\theta_{k,n}}\,\sin^2\theta \cdot \sum_{r=2}^{n-4} \frac{r^2\left(\sqrt{\sin\theta_{k,n}} + \sqrt{\sin\theta}\right)}{n^6\sqrt{r}\,|t_{k,n} - x|^2}\Bigg)$$
$$= O\left(\frac{\sin^2\theta\,(\sin\theta + \sin\theta_{k,n})}{n^3\,|t_{k,n} - x|^2}\right)$$
$$= O\left(\frac{\sin\theta}{n^3\sin^2\frac{|\theta - \theta_{k,n}|}{2}}\right) \text{ for } -1 < t_{k,n} \le \frac{1}{2}.$$

From (3.2.8) and (3.2.11),

(3.3.75) $$|R_2| = O\left(\frac{\sin\theta}{n^2}\right) \text{ for } \frac{1}{2} \le t_{k,n} < 1.$$

We move to R_3. From (3.2.7) and (3.2.11),

(3.3.76) $$|R_3| = O\left(\frac{\sin\theta}{n^2}\right).$$

We next apply (3.2.8), (3.2.11), and Lemma 3.2 (3.3.4) to R_4.

$$|R_4| = O\left(\frac{\sin\theta\,(\sin\theta + \sin\theta_{k,n})}{n^3\,|t_{k,n} - x|}\right) \tag{3.3.77}$$

$$= O\left(\frac{\sin\theta}{n^3 \sin\frac{|\theta - \theta_{k,n}|}{2}}\right) \text{ for } -1 < t_{k,n} \le \frac{1}{2}.$$

Also, from (3.2.8) and (3.2.11),

$$|R_4| = O\left(\frac{\sin\theta}{n^2}\right) \text{ for } \frac{1}{2} \le t_{k,n} < 1. \tag{3.3.78}$$

From (3.2.8), (3.2.11), and Lemma 3.1 (3.3.2),

$$|R_5| = O\left(\frac{\sin^2\theta\sqrt{\sin\theta_{k,n}}\left(\sqrt{\sin\theta_{k,n}} + \sqrt{\sin\theta}\right)}{n^4\,(t_{k,n} - x)^2}\right)$$

$$= O\left(\frac{\sin^2\theta\,(\sin\theta + \sin\theta_{k,n})}{n^4\,|t_{k,n} - x|^2}\right) \tag{3.3.79}$$

$$= O\left(\frac{\sin\theta}{n^3 \sin^2\frac{|\theta - \theta_{k,n}|}{2}}\right) \text{ for } -1 < t_{k,n} \le \frac{1}{2}.$$

When $\frac{1}{2} \le t_{k,n} < 1$, we use (3.2.28) and (3.2.11) to get

$$|R_5| = O\left(\frac{\sin\theta}{n^2}\right). \tag{3.3.80}$$

From (3.2.8) and (3.2.11),

$$(3.3.81)\quad |R_6| = O\left(\frac{\sin\theta}{n^2}\right).$$

Finally, we concentrate our efforts on R_1.

Define

(3.3.82)

$$\gamma_{r,n} = \frac{-2n^2(n-1)^2(2r+3) + 4n(n-1)(r+2)(3r^2+7r+3) - 2r(r+1)(r+2)^2(r+2)}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}\alpha_{r+4,n}}$$

and

$$\sigma_{n-4,n} = \frac{(n-3)^2(n^2+4n-8)}{\alpha_{n-4,n}\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}}.$$

From summation by parts (3.2.1) and Lemma 3.3,

$$\sum_{r=2}^{n-4} \frac{(r+1)^2[2n(n-1) - r(r+2)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}}$$

$$\cdot [P_r'(x) P_{r+1}(t_{k,n}) - P_{r+1}'(x) P_r(t_{k,n})]$$

$$= \sum_{r=2}^{n-5} \left[\frac{(r+1)^2[2n(n-1) - r(r+2)}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}} \right.$$

$$\left. - \frac{(r+2)^2[2n(n-1) - (r+1)(r+3)}{\alpha_{r+1,n}\alpha_{r+2,n}\alpha_{r+3,n}\alpha_{r+4,n}} \right]$$

$$\cdot \sum_{s=2}^{r} \left[P_s'(x) P_{s+1}(t_{k,n}) - P_{s+1}'(x) P_s(t_{k,n}) \right]$$

$$+ \frac{(n-3)^2[2n(n-1) - (n-4)(n-2)]}{\alpha_{n-4,n}\alpha_{n-3,n}\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\cdot \sum_{s=2}^{n-4} [P_s'(x) P_{s+1}(t_{k,n}) - P_{s+1}'(x) P_s(t_{k,n})] \tag{3.3.83}$$

$$= \sum_{r=2}^{n-5} \gamma_{r,n} \left[\frac{r(P_{r-1}(x) - P_r(x))}{1-x} P_r(t_{k,n}) \right.$$

$$, + 1 - P_r'(x) P_r(t_{k,n})$$

$$\left. - r \frac{P_{r-1}(x) P_r(t_{k,n}) - P_r(x) P_{r-1}(t_{k,n})}{t_{k,n} - x} \right]$$

$$+ \sigma_{n-4,n} \left[\frac{(n-4)(P_{n-5}(x) - P_{n-4}(x))}{1-x} \right.$$

$$\cdot P_{n-4}(t_{k,n}) + 1 - P_{n-4}'(x) P_{n-4}(t_{k,n}) - (n-4)$$

$$\left. \cdot \frac{P_{n-5}(x) P_{n-4}(t_{k,n}) - P_{n-4}(x) P_{n-5}(t_{k,n})}{t_{k,n} - x} \right].$$

We may now write

(3.3.84) $R_1 = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$

where

(3.3.85)
$$S_1 = \frac{16n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2\left[P_{n-1}'(t_{k,n})\right]^3(t_{k,n}-x)} \cdot \sum_{r=2}^{n-5} \gamma_{r,n}\left[\frac{r(P_{r-1}(x)-P_r(x))}{1-x}P_r(t_{k,n})+1\right],$$

(3.3.86)
$$S_2 = \frac{-16n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2\left[P_{n-1}'(t_{k,n})\right]^3(t_{k,n}-x)} \cdot \sum_{r=2}^{n-5} \gamma_{r,n}P_r'(x)P_r(t_{k,n}),$$

(3.3.87)
$$S_3 = \frac{16n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2\left[P_{n-1}'(t_{k,n})\right]^3(t_{k,n}-x)^2} \cdot \sum_{r=2}^{n-5} \gamma_{r,n}r\left[P_{r-1}(x)P_r(t_{k,n}) - P_r(x)P_{r-1}(t_{k,n})\right],$$

(3.3.88)
$$S_4 = \frac{16n(n-1)(1-x^2)P_{n-1}(x)\sigma_{n-4,n}}{(1-t_{k,n}^2)^2\left[P_{n-1}'(t_{k,n})\right]^3(t_{k,n}-x)} \cdot \left[\frac{(n-4)(P_{n-5}(x)-P_{n-4}(x))}{1-x}P_{n-4}(t_{k,n})+1\right],$$

(3.3.89)
$$S_5 = \frac{-16n(n-1)(1-x^2)P_{n-1}(x)\sigma_{n-4,n}}{(1-t_{k,n}^2)^2\left[P_{n-1}'(t_{k,n})\right]^3(t_{k,n}-x)} \cdot \left[P_{n-4}'(x)P_{n-4}(t_{k,n})\right],$$

and

$$S_6 = \frac{-16n(n-1)(1-x^2)P_{n-1}(x)\,\sigma_{n-4,n}}{(1-t_{k,n}^2)^2\left[P'_{n-1}(t_{k,n})\right]^3(t_{k,n}-x)^2} \tag{3.3.90}$$
$$\cdot\left[P_{n-5}(x)P_{n-4}(t_{k,n}) - P_{n-4}(x)P_{n-5}(t_{k,n})\right].$$

We now estimate the order of the terms $S_1 - S_6$. To begin, notice that $|\gamma_{r,n}| = O\left(\frac{1}{n^5}\right)$ and $|\sigma_{n-4,n}| = O\left(\frac{1}{n^4}\right)$. From (3.2.7), (3.2.8), and (3.2.11),

$$|S_1| = O\left(\frac{\sin^{3/2}\theta\sqrt{\sin\theta_{k,n}}}{|t_{k,n}-x|}\sum_{r=2}^{n-5}\frac{r}{n^5}\right) \tag{3.3.91}$$
$$= O\left(\frac{\sin\theta}{n^3\sin\frac{|\theta-\theta_{k,n}|}{2}}\right).$$

Using (3.2.7), (3.2.8), (3.2.10), and (3.2.11), one may obtain

$$|S_2| = O\left(\frac{\sqrt{n}\,\sin^2\theta\sqrt{\sin\theta_{k,n}}}{|t_{k,n}-x|}\sum_{r=2}^{n-5}\frac{\sqrt{r}}{n^5\sin\theta\sqrt{\sin\theta_{k,n}}}\right) \tag{3.3.92}$$
$$= O\left(\frac{\sin\theta}{n^3\sin^2\frac{|\theta-\theta_{k,n}|}{2}}\right).$$

We apply (3.2.8), (3.2.11), and Lemma 3.1 (3.3.2), to derive

$$|S_3| = O\left(\frac{\sqrt{n}\,\sin^2\theta\sqrt{\sin\theta_{k,n}}}{|t_{k,n} - x|^2}\sum_{r=2}^{n-5}\frac{(\sqrt{\sin\theta} + \sqrt{\sin\theta_{k,n}})\,\sqrt{r}}{n^5}\right)$$

(3.3.93)

$$= O\left(\frac{\sin\theta}{n^3\sin^2\frac{|\theta - \theta_{k,n}|}{2}}\right) \text{ for } -1 < t_{k,n} \le \frac{1}{2}.$$

From (3.2.7), (3.2.8), and (3.2.11),

(3.3.94) $$|S_3| = O\left(\frac{\sin\theta}{n^2}\right) \text{ for } \frac{1}{2} \le t_{k,n} < 1.$$

We use estimates (3.2.7), (3.2.8), and (3.2.11), to write

$$|S_4| = O\left(\frac{\sin^2\theta}{n^3|t_{k,n} - x|}\right)$$

(3.3.95)

$$= O\left(\frac{\sin\theta}{n^3\sin\frac{|\theta - \theta_{k,n}|}{2}}\right).$$

From (3.2.7), (3.2.8), (3.2.10), and (3.2.11),

$$|S_5| = O\left(\frac{\sin\theta}{n^3|t_{k,n} - x|}\right)$$

(3.3.96)

$$= O\left(\frac{\sin\theta}{n^3\sin^2\frac{|\theta - \theta_{k,n}|}{2}}\right).$$

Using (3.2.8), (3.2.11), and Lemma 3.1 (3.3.2), we have

$$
(3.3.97)\qquad
\begin{aligned}
|S_6| &= O\left(\frac{\sin^2\theta\sqrt{\sin\theta_{k,n}}\,(\sqrt{\sin\theta}+\sqrt{\sin\theta_{k,n}})}{n^3\,|t_{k,n}-x|^2}\right)\\
&= O\left(\frac{\sin\theta}{n^3\sin^2\frac{|\theta-\theta_{k,n}|}{2}}\right)\ \text{for } -1 < t_{k,n} \le \frac{1}{2}.
\end{aligned}
$$

Estimates (3.2.8) and (3.2.11) imply that

$$
(3.3.98)\qquad |S_6| = O\left(\frac{\sin\theta}{n^2}\right)\ \text{for } \frac{1}{2} \le t_{k,n} < 1.
$$

The lemma now follows.

<u>Lemma 3.8</u>

$$
(3.3.99)\qquad
\begin{aligned}
&|x - t_{k,n}|\,|r_{k,n}(x)| = O\left(\frac{\sin\theta}{n}\right)\\
&\text{for } |\theta_{k,n}-\theta| < \frac{c}{n},\ k = 2, 3, \ldots, n-1.
\end{aligned}
$$

<u>Proof</u>: We begin by showing that $r_{k,n}(x)$ can be written in the form

$$
(3.3.100)\qquad
\begin{aligned}
r_{k,n}(x) &= \frac{(1-x^2)P_{n-1}^2(x)}{(1-t_{k,n}^2)(x-t_{k,n})^2[P'_{n-1}(t_{k,n})]^2}\\
&- \frac{(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}\\
&\cdot \sum_{r=2}^{n-1}\left[\frac{n(n-1)-r(r-1)}{n(n-1)+r(r-1)}\right]\\
&\cdot (2r-1)P_{r-1}(t_{k,n})P'_{r-1}(x).
\end{aligned}
$$

Notice that from (2.2.4) and (2.2.10),

$$
\begin{aligned}
&\sum_{r=2}^{n-1}\left[\frac{n(n-1)-r(r-1)}{n(n-1)+r(r-1)}\right] \\
&\quad\cdot (2r-1)\,P_{r-1}(t_{k,n})\,P'_{r-1}(x) \\
&\quad= -\sum_{r=2}^{n-1}(2r-1)\,P_{r-1}(t_{k,n})\,P'_{r-1}(x) \\
&\quad+ 2n(n-1)\sum_{r=2}^{n-1}\frac{(2r-1)}{n(n-1)+r(r-1)} \\
&\quad\cdot P_{r-1}(t_{k,n})\,P'_{r-1}(x) = (n-1) \\
&\quad\cdot\left[\frac{P'_{n-1}(x)\,P_{n-2}(t_{k,n})}{t_{k,n}-x}+\frac{P_{n-1}(x)\,P_{n-2}(t_{k,n})}{(t_{k,n}-x)^2}\right] \\
&\quad+ 2n(n-1)\sum_{r=2}^{n-1}\frac{(2r-1)}{n(n-1)+r(r-1)} \\
&\quad\cdot P_{r-1}(t_{k,n})\,P'_{r-1}(x) \\
&\quad= \frac{(1-t_{k,n}^2)\,P'_{n-1}(t_{k,n})\,P'_{n-1}(x)}{t_{k,n}-x} \\
&\quad+ \frac{(1-t_{k,n}^2)\,P'_{n-1}(t_{k,n})\,P_{n-1}(x)}{(t_{k,n}-x)^2} \\
&\quad+ 2n(n-1)\sum_{r=2}^{n-1}\frac{(2r-1)}{n(n-1)+r(r-1)} \\
&\quad\cdot P_{r-1}(t_{k,n})\,P'_{r-1}(x)\,.
\end{aligned}
\tag{3.3.101}
$$

Hence,

$$
\begin{aligned}
r_{k,n}(x) &= \frac{(1-x^2)\,P_{n-1}^2(x)}{(1-t_{k,n}^2)(x-t_{k,n})^2\,[P_{n-1}'(t_{k,n})]^2} \\
&- \frac{(1-x^2)\,P_{n-1}(x)}{(1-t_{k,n}^2)^2\,[P_{n-1}'(t_{k,n})]^3} \\
&\cdot \left\{ \frac{(1-t_{k,n}^2)\,P_{n-1}'(t_{k,n})\,P_{n-1}'(x)}{t_{k,n}-x} \right. \\
&+ \frac{(1-t_{k,n}^2)\,P_{n-1}'(t_{k,n})\,P_{n-1}(x)}{(t_{k,n}-x)^2} + 2n(n-1) \\
&\left. \cdot \sum_{r=2}^{n-1} \frac{(2r-1)}{n(n-1)+r(r-1)}\,P_{r-1}(t_{k,n})\,P_{r-1}'(x) \right\} \\
&= \frac{(1-x^2)\,P_{n-1}(x)\,P_{n-1}'(x)}{(1-t_{k,n}^2)(x-t_{k,n})\,[P_{n-1}'(t_{k,n})]^2} \\
&- \frac{2n(n-1)(1-x^2)\,P_{n-1}(x)}{(1-t_{k,n}^2)^2\,[P_{n-1}'(t_{k,n})]^3} \\
&\cdot \sum_{r-2}^{n-1} \frac{(2r-1)\,P_{r-1}(t_{k,n})\,P_{r-1}'(x)}{n(n-1)+r(r-1)}\,.
\end{aligned}
\tag{3.3.102}
$$

From (2.1.12), we see (3.3.100) is correct. Next, we calculate the order of $|x - t_{k,n}|\,|r_{k,n}(x)|$. Recalling (3.2.2) - (3.2.5), (3.2.7), and (3.2.19), we state

$$\frac{(1 - x^2)\, P_{n-1}^2(x)}{(1 - t_{k,n}^2)\, |x - t_{k,n}|\, [P_{n-1}'(t_{k,n})]^2}$$

$$(3.3.103) \qquad = O\left(\frac{(1 - x^2)\, P_{n-1}(x)}{(1 - t_{k,n}^2)\, P_{n-1}'(t_{k,n})}\right)$$

$$= O\left(\frac{\sin\theta\, \sqrt{\sin\theta}}{n\, \sqrt{\sin\theta_{k,n}}}\right).$$

From (3.2.2), (3.2.3), and $|\theta - \theta_{k,n}| < \frac{c}{n}$ we have

$$\frac{\sin\theta}{\sin\theta_{k,n}} = 1 + \frac{\sin\theta - \sin\theta_{k,n}}{\sin\theta_{k,n}}$$

$$(3.3.104) \qquad = 1 + \frac{2\cos\frac{\theta + \theta_{k,n}}{2}\, \sin\frac{|\theta - \theta_{k,n}|}{2}}{\sin\theta_{k,n}}$$

$$\leq 1 + \frac{2\sin\frac{|\theta - \theta_{k,n}|}{2}}{\sin\theta_{k,n}} = O\,(1)\,.$$

From (3.3.103) and (3.3.104),

$$(3.3.105) \qquad \frac{(1 - x^2)\, P_{n-1}^2(x)}{(1 - t_{k,n}^2)\, |x - t_{k,n}|\, [P_{n-1}'(t_{k,n})]^2} = O\left(\frac{\sin\theta}{n}\right).$$

Recalling that $|\theta - \theta_{k,n}| < \frac{c}{n}$, we have

$$|x - t_{k,n}| = \sin\frac{\theta + \theta_{k,n}}{2}\,\sin\frac{|\theta - \theta_{k,n}|}{2}$$

$$\le \left[\sin\theta\cos\frac{|\theta_{k,n} - \theta|}{2} + |\cos\theta|\sin\frac{|\theta_{k,n} - \theta|}{2}\right]\frac{1}{n} \tag{3.3.106}$$

$$= O\left(\frac{\sin\theta}{n} + \frac{1}{n^2}\right).$$

It follows from (3.2.7), (3.2.9), and (3.2.11) that

$$\frac{(1 - x^2)\,|P_{n-1}(x)|}{(1 - t_{k,n}^2)^2\,|P'_{n-1}(t_{k,n})|^3} \cdot \sum_{r=2}^{n-1}\left[\frac{n(n-1) - r(r-1)}{n(n-1) + r(r-1)}\right] \cdot (2r - 1)\,|P_{r-1}(t_{k,n})\,P'_{r-1}(x)| = O(1). \tag{3.3.107}$$

Using (3.2.7), (3.2.8), (3.2.10), and (3.2.11), we obtain an alternative estimate.

$$\frac{(1 - x^2)\,|P_{n-1}(x)|}{(1 - t_{k,n}^2)^2\,|P'_{n-1}(t_{k,n})|^3} \cdot \sum_{r=2}^{n-1}\left[\frac{n(n-1) - r(r-1)}{n(n-1) + r(r-1)}\right] \cdot (2r - 1)\,|P_{r-1}(t_{k,n})\,P'_{r-1}(x)| = O(n\sin\theta). \tag{3.3.108}$$

From (3.3.105) - (3.3.108), the lemma follows.

Lemma 3.9

(3.3.109) $\quad |(1 - x)\, r_{n,n}(x)| = O\left(\frac{\sin\theta}{n}\right)$ for $-1 < x < 1$.

Proof: Using (3.2.7),

(3.3.110) $\quad \frac{(1 - x^2)}{2}\, |P_{n-1}(x)\, P_{n-2}(x)| = O\left(\frac{\sin\theta}{n}\right).$

Define

$$k(x) = (1 - x)(1 - x^2)\, P_{n-1}(x)$$

(3.3.111)
$$\cdot \sum_{r=2}^{n-2} \frac{(2r - 1)\, P'_{r-1}(x)}{n(n - 1) + r(r - 1)}$$

$$+ \frac{(n - 2)}{2(n - 1)^2}\, (1 - x)(1 - x^2)\, P_{n-1}(x)\, P'_{n-2}(x)\,.$$

Recalling (2.1.13), we see that it is left to show

(3.3.112) $\quad |k(x)| = O\left(\frac{\sin\theta}{n}\right).$

As in Lemma 3.7, define

(3.3.113) $\quad \alpha_{r,n} = n(n - 1) + r(r - 1)\,.$

We next apply (3.2.1) to the following

$$\sum_{r=2}^{n-2} \frac{(2r-1)P'_{r-1}(x)}{n(n-1)+r(r-1)} = \sum_{r=2}^{n-3}\left[\frac{1}{\alpha_{r,n}} - \frac{1}{\alpha_{r+1,n}}\right]$$

$$\cdot \sum_{s=2}^{r} (2s-1)P'_{s-1}(x) + \frac{1}{n(n-1)+(n-2)(n-3)}$$

$$(3.3.114) \qquad \cdot \sum_{s=2}^{n-2} (2s-1)P'_{s-1}(x) = \sum_{r=2}^{n-3} \frac{2r}{\alpha_{r,n}\alpha_{r+1,n}}$$

$$\cdot \sum_{s=2}^{r} (2s-1)P'_{s-1}(x)P_{s-1}(1) + \frac{1}{2(n^2-3n+3)}$$

$$\cdot \sum_{s=2}^{n-2} (2s-1)P'_{s-1}(x)P_{s-1}(1)\,.$$

From (2.2.8) and (2.2.10),

$$\sum_{s=2}^{r} (2s-1)P'_{s-1}(x)P_{s-1}(1)$$

$$= r\left[\frac{P'_{r-1}(x)-P'_r(x)}{1-x} + \frac{P_{r-1}(x)-P_r(x)}{(1-x)^2}\right]$$

$$(3.3.115)$$

$$= r\,\frac{P'_{r-1}(x)-P'_r(x)}{1-x} + \frac{P'_{r-1}(x)+P'_r(x)}{1-x}$$

$$= (r+1)\,\frac{P'_{r-1}(x)-P'_r(x)}{(1-x)} + \frac{2P'_r(x)}{(1-x)}\,.$$

Hence,

$$(3.3.116) \qquad k(x) = A_1 + A_2 + A_3 + A_4 + A_5$$

where

$$A_1 = (1 - x^2)\, P_{n-1}(x)$$

(3.3.117)
$$\cdot \sum_{r=2}^{n-3} \frac{2r(r+1)}{\alpha_{r,n}\alpha_{r+1,n}} \, [P'_{r-1}(x) \;\; P'_r(x)] \,,$$

$$A_2 = (1 - x^2)\, P_{n-1}(x)$$

(3.3.118)
$$\cdot \sum_{r=2}^{n-3} \frac{4r}{\alpha_{r,n}\alpha_{r+1,n}} \, P'_r(x) \,,$$

$$A_3 = \frac{(n-1)}{2(n^2 - 3n + 3)} \, (1 - x^2)$$

(3.3.119)
$$\cdot P_{n-1}(x) \;\; [P'_{n-3}(x) - P'_{n-2}(x)] \,,$$

(3.3.120) $$A_4 = \frac{1}{(n^2 - 3n + 3)} \, (1 - x^2)\, P_{n-1}(x)\, P'_{n-2}(x) \,,$$

and

(3.3.121) $$A_5 = \frac{(n-2)}{2(n-1)^2} \, (1 - x)(1 - x^2)\, P_{n-1}(x)\, P'_{n-2}(x) \,.$$

From (3.2.8) and (3.2.10),

(3.3.122) $$|A_2| = O\left(\sin^2\theta \sum_{r=2}^{n-3} \frac{r^2}{n^4 \sin\theta}\right) = O\left(\frac{\sin\theta}{n}\right)$$

and

(3.3.123) $$|A_4| = O\left(\frac{\sin\theta}{n}\right).$$

Using (3.2.1) and the observation that we have a telescoping series,

$$\sum_{r=2}^{n-3} \frac{r(r+1)}{\alpha_{r,n}\alpha_{r+1,n}} [P'_{r-1}(x) - P'_r(x)]$$

(3.3.124)

$$= \sum_{r=2}^{n-4} \left[\frac{r(r+1)}{\alpha_{r,n}\alpha_{r+1,n}} - \frac{(r+1)(r+2)}{\alpha_{r+1,n}\alpha_{r+2,n}} \right] \cdot \sum_{s=2}^{r} [P'_{s-1}(x) - P'_s(x)] + \frac{(n-3)(n-2)}{\alpha_{n-3,n}\alpha_{n-2,n}} \cdot \sum_{s=2}^{n-3} [P'_{s-1}(x) - P'_s(x)]$$

$$= -\sum_{r=2}^{n-4} \frac{2(r+1)[n(n-1) - r(r+2)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}} \cdot [1 - P'_r(x)] + \frac{(n-3)(n-2)}{\alpha_{n-3,n}\alpha_{n-2,n}} [1 - P'_{n-3}(x)] .$$

Together (3.2.8) and (3.2.10) along with (3.3.117) and (3.3.124) imply

(3.3.125) $$|A_1| = O\left(\sin^2\theta \sum_{r=2}^{n-4} \frac{r}{n^3 \sin\theta} + \frac{\sin\theta}{n}\right) = O\left(\frac{\sin\theta}{n}\right).$$

Lastly,

(3.3.126)

$$A_3 + A_5 = \frac{(n-1)}{2(n^2 - 3n + 3)} (1 - x^2) P_{n-1}(x) \cdot \left[P'_{n-3}(x) - P'_{n-2}(x) + (1 - x) P'_{n-2}(x)\right] + \frac{(-2n^2 + 6n - 5)}{2(n-1)^2(n^2 - 3n + 3)} (1 - x^2)(1 - x) P_{n-1}(x) P'_{n-2}(x) .$$

The estimates (3.2.8) and (3.2.10) imply that

$$\left| \frac{(-2n^2 + 6n - 5)}{2(n-1)^2(n^2 - 3n + 3)} (1 - x^2)(1 - x) P_{n-1}(x) \cdot P'_{n-2}(x) \right| = O\left(\frac{\sin\theta}{n}\right). \tag{3.3.127}$$

Using (2.2.5), we obtain

$$\begin{aligned} & P'_{n-3}(x) - P'_{n-2}(x) + (1 - x) P'_{n-2}(x) \\ & = P'_{n-3}(x) - xP'_{n-2}(x) \\ & = -(n - 2) P_{n-2}(x). \end{aligned} \tag{3.3.128}$$

Hence, from (3.3.128) and estimate (3.2.7),

$$\left| \frac{(n-1)}{2(n^2 - 3n + 3)} (1 - x^2) P_{n-1}(x) \left[P'_{n-3}(x) - P'_{n-2}(x) + (1 - x) P'_{n-2}(x)\right] \right| = O\left(\frac{\sin\theta}{n}\right). \tag{3.3.129}$$

We conclude that

$$|A_3 + A_5| = O\left(\frac{\sin\theta}{n}\right), \tag{3.3.130}$$

and the proof is completed.

Lemma 3.10

$$\sum_{k=0}^{n} |x - t_{k,n}|\, |r_{k,n}(x)| = O\left(\frac{\sin\theta}{n}\right) \text{ for } -1 \le x \le 1. \tag{3.3.131}$$

<u>Proof</u>: We begin by noting that we need only prove the result for $-1 < x < 1$ since $(1 + x)r_{o,n}(x)$ and $(1 - x)r_{n,n}(x)$ vanish at $x = \pm 1$, and $r_{k,n}(\pm 1) = 0$ for $k = 1, 2, \ldots, n - 1$.

We now look at the terms $|1 + x|\,|r_{o,n}(x)|$ and $|1 - x|\,|r_{n,n}(x)|$. From Lemma 3.9,

$$|1 - x|\,|r_{n,n}(x)| = O\left(\frac{\sin\theta}{n}\right) \text{ for } -1 < x < 1 . \tag{3.3.132}$$

Since $r_{n,n}(-x) = r_{o,n}(x)$ from (2.1.13), we have

$$\begin{aligned} |1 + x|\,|r_{o,n}(x)| &= |1 - (-x)|\,|r_{n,n}(-x)| \\ &= O\left(\frac{\sin\theta}{n}\right) \text{ for } -1 < x < 1 . \end{aligned} \tag{3.3.133}$$

Hence, it is left to show that

$$\sum_{k=1}^{n-1} |x - t_{k,n}|\,|r_{k,n}(x)| = O\left(\frac{\sin\theta}{n}\right) \text{ for } -1 < x < 1 . \tag{3.3.134}$$

Actually, we shall see that we need only prove the above inequality for $-1 \le x < 0$.

It follows from symmetry and uniqueness of the $r_{k,n}(x)$'s, that

$$r_{k,n}(-x) = r_{n-k,n}(x) \text{ and } t_{k,n} = -t_{n-k,n} \text{ for } k = 1, 2, \ldots, n - 1 . \tag{3.3.135}$$

Further,

$$\sum_{k=1}^{n-1} |(-x) - t_{k,n}|\,|r_{k,n}(-x)|$$

(3.3.136)
$$= \sum_{k=1}^{n-1} |x - t_{n-k,n}|\,|r_{n-k,n}(x)|$$

$$= \sum_{k=1}^{n-1} |x - t_{k,n}|\,|r_{k,n}(x)|\,.$$

Therefore, proving (3.3.134) for $-1 < x \le 0$ is sufficient. To begin,

$$\sum_{k=1}^{n-1} |x - t_{k,n}|\,|r_{k,n}(x)|$$

(3.3.137)
$$= \sum_{|\theta-\theta_{k,n}| < \frac{c}{n}} |x - t_{k,n}|\,|r_{k,n}(x)|$$

$$+ \sum_{|\theta-\theta_{k,n}| \ge \frac{c}{n}} |x - t_{k,n}|\,|r_{k,n}(x)|\,.$$

From (3.2.6), there are a finite number of different $\theta_{k,n}$'s such that $|\theta - \theta_{k,n}| < \frac{c}{n}$. From Lemma 3.8, we may conclude that

(3.3.138)
$$\sum_{|\theta-\theta_{k,n}| < \frac{c}{n}} |x - t_{k,n}|\,|r_{k,n}(x)| = O\left(\frac{\sin\theta}{n}\right)$$

From Lemma 3.7,

$$\sum_{|\theta-\theta_{k,n}| \ge \frac{c}{n}} |x - t_{k,n}| \, |r_{k,n}(x)|$$

$$(3.3.139) \qquad = O\left(\sum_{|\theta-\theta_{k,n}| \ge \frac{c}{n}} \frac{\sin\theta}{n} \left[\frac{1}{n^2 \sin^2 \frac{|\theta-\theta_{k,n}|}{2}} + \frac{1}{n^3 \sin^3 \frac{|\theta-\theta_{k,n}|}{2}} + \frac{1}{n} \right] \right)$$

After applying (3.2.17) and (3.2.18) to (3.3.139), we have

$$(3.3.140) \qquad \sum_{|\theta-\theta_{k,n}| \ge \frac{c}{n}} |x - t_{k,n}| \, |r_{k,n}(x)| = O\left(\frac{\sin\theta}{n}\right)$$

The proof of the lemma is now complete.

Lemma 3.11

For $-1 \le x \le 1$,

$$(3.3.141) \qquad \sum_{k=0}^{n} |r_{k,n}(x)| = O(1) \, .$$

Proof: From the definition of $r_{k,n}(x)$ in (2.1.13) and the estimates (3.2.7), (3.2.8), (3.2.9), and (3.2.11), we have

$$(3.3.142) \qquad |r_{n,n}(x)| = |r_{o,n}(-x)| = O(1) \text{ for } -1 < x < 1 .$$

Note that because $\sum_{k=0}^{n} |r_{k,n}(\pm 1)| = 1$, it is sufficient to prove (3.3.141) on the open interval $(-1,1)$.

Following an argument similar to (3.3.136), we have

$$(3.3.143)\quad \sum_{k=1}^{n-1} |r_{k,n}(-x)| = \sum_{k=1}^{n-1} |r_{k,n}(x)|.$$

We are, therefore, left with the task of showing

$$(3.3.144)\quad \sum_{k=1}^{n-1} |r_{k,n}(x)| = O(1) \text{ for } -1 < x \le 0.$$

From Lemma 3.8 (3.3.100), we may write

$$(3.3.145)\quad r_{k,n}(x) = C_{k,n} - D_{k,n}$$

where

$$C_{k,n} = \frac{(1 - x^2)\, P_{n-1}^2(x)}{(1 - t_{k,n}^2)\,(x - t_{k,n})^2\,[P_{n-1}'(t_{k,n})]^2}$$

and

$$D_{k,n} = \frac{(1 - x^2)\, P_{n-1}(x)}{(1 - t_{k,n}^2)^2\,[P_{n-1}'(t_{k,n})]^3}$$

$$\cdot \sum_{r=2}^{n-1} \left[\frac{n(n-1) - r(r-1)}{n(n-1) + r(r-1)}\right]$$

$$\cdot\,(2r - 1)\, P_{r-1}(t_{k,n})\, P_{r-1}'(x)\,.$$

A result of Turán [11] states that

$$\sum_{k=1}^{n-1} \frac{(1 - x^2)\, P_{n-1}^2(x)}{(1 - t_{k,n}^2)\,(x - t_{k,n})^2\,[P_{n-1}'(t_{k,n})]^2} \tag{3.3.146}$$

$$= 1 - P_{n-1}^2(x) = O(1)\,.$$

Hence, we need only show that

$$\sum_{k=1}^{n-1} |D_{k,n}| = O\,(1) \text{ for } -1 < x \le 0, \tag{3.3.147}$$

and the lemma is proved. We now restrict x to the interval [-1,1].

From (3.3.107),

$$|D_{k,n}| = O\,(1)\,. \tag{3.3.148}$$

We proceed to obtain an alternative estimate of $D_{k,n}$. Recall that we earlier defined

$$\alpha_{r,n} = n(n-1) + r(r-1)\,. \tag{3.3.149}$$

From summation by parts (3.2.1) and (2.2.10), we obtain

$$D_{k,n} = \frac{(1 - x^2)\, P_{n-1}(x)}{(1 - t_{k,n}^2)^2\,[P_{n-1}'(t_{k,n})]^3}$$

$$\cdot \left\{ \sum_{r=2}^{n-2} \left[\frac{n(n-1) - r(r-1)}{n(n-1) + r(r-1)} - \frac{n(n-1) - r(r+1)}{n(n-1) + r(r-1)} \right] \right. \tag{3.3.150}$$

$$\cdot \sum_{s=2}^{r} (2s-1)\, P_{s-1}(t_{k,n})\, P_{s-1}'(x) + \left[\frac{n(n-1) - (n-1)(n-2)}{n(n-1) + (n-1)(n-2)} \right]$$

$$\left. \cdot \sum_{s=2}^{n-1} (2s-1)\, P_{s-1}(t_{k,n})\, P_{s-1}'(x) \right\} = S_{1,k} + S_{2,k} + S_{3,k} + S_{4,k}$$

where

$$S_{1,k} = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3(t_{k,n}-x)}$$

$$\cdot \sum_{r=2}^{n-2} \frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}} \left[P'_{r-1}(x)P_r(t_{k,n}) - P'_r(x)P_{r-1}(t_{k,n})\right],$$

$$S_{2,k} = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3(t_{k,n}-x)^2}$$

$$\cdot \sum_{r=2}^{n-2} \frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}} \left[P_{r-1}(x)P_r(t_{k,n}) - P_r(x)P_{r-1}(t_{k,n})\right],$$

$$S_{3,k} = \frac{-(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}\left[\frac{P'_{n-1}(x)P_{n-2}(t_{k,n})}{t_{k,n}-x}\right],$$

$$S_{4,k} = \frac{-(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P'_{n-1}(t_{k,n})]^3}\left[\frac{P_{n-1}(x)P_{n-2}(t_{k,n})}{(t_{k,n}-x)^2}\right].$$

From (3.2.7), (3.2.11), and Lemma 3.1 (3.3.1),

$$|S_{2,k}| = O\left(\frac{\sin\theta}{|t_{k,n} - x|^2} \sum_{r=2}^{n-2} \frac{r(\sin\theta + \sin\theta_{k,n})}{n^4}\right)$$

(3.3.151)

$$= O\left(\frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}}\right) \text{ for } -1 < t_{k,n} \le \frac{1}{2}.$$

We use estimates (3.2.7), (3.2.8), and (3.2.11) to get

$$(3.3.152)\qquad |S_{2,k}| = O\left(\frac{1}{n}\right) \text{ for } \frac{1}{2} \le t_{k,n} < 1.$$

From (3.2.7), (3.2.9), (3.2.11), and (3.2.16),

$$(3.3.153)\qquad |S_{3,k}| = O\left(\frac{\sin\theta_{k,n}}{n^2|t_{k,n} - x|}\right) = O\left(\frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}}\right).$$

Using (3.2.8), (3.2.11), and (3.2.16), we obtain

$$(3.3.154)\qquad |S_{4,k}| = O\left(\frac{\sin^2\theta \sin\theta_{k,n}}{n^2|t_{k,n} - x|^2}\right) = O\left(\frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}}\right).$$

Finally, we estimate the order of $S_{1,k}$. From (3.2.1) and Lemma 3.3,

$$S_{1,k} = \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3(t_{k,n}-x)}$$

$$\cdot\left\{\sum_{r=2}^{n-3}\left[\frac{r^2}{\alpha_{r,n}\alpha_{r+1,n}} - \frac{(r+1)^2}{\alpha_{r+1,n}\alpha_{r+2,n}}\right]\right.$$

$$\cdot\sum_{s=2}^{r}\left[P_{s-1}'(x)P_s(t_{k,n}) - P_s'(x)P_{s-1}(t_{k,n})\right]$$

$$\left. + \frac{(n-2)^2}{\alpha_{n-2,n}\alpha_{n-1,n}}\sum_{s=2}^{n-2}\left[P_{s-1}'(x)P_s(t_{k,n}) - P_s'(x)P_{s-1}(t_{k,n})\right]\right\}$$

$$= \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3(t_{k,n}-x)}$$

$$(3.3.155) \qquad \cdot\left\{\sum_{r=2}^{n-3}\frac{(2r+1)[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}\right.$$

$$\cdot\left[\frac{r(P_{r-1}(x)-P_r(x))}{1-x}P_r(t_{k,n}) + 1 - P_r'(x)P_r(t_{k,n})\right.$$

$$\left. - r\frac{P_{r-1}(x)P_r(t_{k,n}) - P_r(x)P_{r-1}(t_{k,n})}{t_{k,n}-x}\right] + \frac{(n-2)^2}{\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\cdot\left[\frac{(n-2)(P_{n-3}(x)-P_{n-2}(x))}{1-x}P_{n-2}(t_{k,n})\right.$$

$$+ 1 - P_{n-2}'(x)P_{n-2}(t_{k,n}) - (n-2)$$

$$\left.\left.\cdot\frac{P_{n-3}(x)P_{n-2}(t_{k,n}) - P_{n-2}(x)P_{n-3}(t_{k,n})}{t_{k,n}-x}\right]\right\}.$$

We use estimates (3.2.7) and (3.2.11) to get

$$\left| \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3(t_{k,n}-x)} \right.$$

$$\cdot \left\{ \sum_{r=2}^{n-3} \frac{(2r+1)[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}} \right.$$

$$\cdot \left[\frac{r(P_{r-1}(x)-P_r(x))}{1-x} P_r(t_{k,n}) + 1 \right]$$

(3.3.156)

$$+ \frac{(n-2)^2}{\alpha_{n-2,n}\alpha_{n-1,n}}$$

$$\left. \left. \cdot \left[\frac{(n-2)(P_{n-3}(x)-P_{n-2}(x))}{1-x} P_{n-2}(t_{k,n}) + 1 \right] \right\} \right|$$

$$= O\left(\frac{\sin\theta}{n^2|t_{k,n}-x|} \right) = O\left(\frac{1}{n^2\sin^2\frac{|\theta-\theta_{k,n}|}{2}} \right).$$

From (3.2.7), (3.2.9), and (3.2.11),

$$\left| \frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3(t_{k,n}-x)} \right.$$

$$\cdot \left\{ \sum_{r=2}^{n-3} \frac{(2r+1)[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}} \right.$$

(3.3.157)

$$\left. \left. \cdot P_r'(x)P_r(t_{k,n}) + \frac{(n-2)^2P_{n-2}'(x)P_{n-2}(t_{k,n})}{\alpha_{n-2,n}\alpha_{n-1,n}} \right\} \right|$$

$$= O\left(\frac{1}{n^2|t_{k,n}-x|^2} \right) = O\left(\frac{1}{\sin^2\frac{|\theta-\theta_{k,n}|}{2}} \right).$$

We next use (3.2.7), (3.2.11), and Lemma 3.1 (3.3.1) to get

(3.3.158)
$$\left|\frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3(t_{k,n}-x)^2}\right.$$
$$\cdot\left\{\sum_{r=2}^{n-3}\frac{(2r+1)\,r\,[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}\right.$$
$$\cdot[P_{r-1}(x)P_r(t_{k,n})-P_r(x)P_{r-1}(t_{k,n})]$$
$$+\frac{(n-2)^3}{\alpha_{n-2,n}\alpha_{n-1,n}}[P_{n-3}(x)P_{n-2}(t_{k,n})$$
$$\left.\left.-P_{n-2}(x)P_{n-3}(t_{k,n})]\right\}\right|$$
$$=O\left(\frac{\sin\theta(\sin\theta+\sin\theta_{k,n})}{n^2|t_{k,n}-x|^2}\right)$$
$$=O\left(\frac{1}{n^2\sin^2\frac{|\theta-\theta_{k,n}|}{2}}\right)\text{ for } -1<t_{k,n}\le\frac{1}{2}.$$

From (3.2.7) and (3.2.11),

(3.3.159)
$$\left|\frac{4n(n-1)(1-x^2)P_{n-1}(x)}{(1-t_{k,n}^2)^2[P_{n-1}'(t_{k,n})]^3(t_{k,n}-x)^2}\right.$$
$$\cdot\left\{\sum_{r=2}^{n-3}\frac{(2r+1)\,r\,[n(n-1)-r(r+1)]}{\alpha_{r,n}\alpha_{r+1,n}\alpha_{r+2,n}}\right.$$
$$\cdot[P_{r-1}(x)P_r(t_{k,n})-P_r(x)P_{r-1}(t_{k,n})]$$
$$+\frac{(n-2)^3}{\alpha_{n-2,n}\alpha_{n-1,n}}[P_{n-3}(x)P_{n-2}(t_{k,n})$$
$$\left.\left.-P_{n-2}(x)P_{n-3}(t_{k,n})]\right\}\right|$$
$$=O\left(\frac{1}{n}\right)\text{ for }\frac{1}{2}\le t_{k,n}<1.$$

It follows from (3.3.155) - (3.3.159) that

$$(3.3.160) \quad |S_{1,k}| = O\left(\frac{1}{n} + \frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}}\right) \text{ for } |\theta - \theta_{k,n}| \geq \frac{c}{n}.$$

Hence,

$$(3.3.161) \quad |D_{k,n}| = O\left(\frac{1}{n} + \frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}}\right).$$

We now break into two different sums as follows.

$$(3.3.162) \quad \sum_{k=1}^{n-1} |D_{k,n}| = \sum_{|\theta-\theta_{k,n}|<\frac{c}{n}} |D_{k,n}| + \sum_{|\theta-\theta_{k,n}|\geq\frac{c}{n}} |D_{k,n}|$$

From (3.2.6), the first sum on the right hand side of (3.3.162) contains a finite number of terms. Hence, (3.3.147) implies that

$$(3.3.163) \quad \sum_{|\theta-\theta_{k,n}|<\frac{c}{n}} |D_{k,n}| = O\,(1).$$

From (3.3.161) and (3.2.17),

$$(3.3.164) \quad \sum_{|\theta-\theta_{k,n}|\geq\frac{c}{n}} |D_{k,n}| = \sum_{|\theta-\theta_{k,n}|\geq\frac{c}{n}} \left(\frac{1}{n} + \frac{1}{n^2 \sin^2 \frac{|\theta - \theta_{k,n}|}{2}}\right) = O\,(1).$$

The proof of the lemma is now complete.

Proof of Convergence Results

We begin this section with a result of DeVore, Theorem 2.4 in [10]. Let Lip 1 denote the class of Lipschitz one functions and C[-1,1] the class of continuous functions on the interval [-1,1].

Theorem 3.A

Suppose that L_n is a bounded linear operator on C[-1,1]. If $C_1 \geq 1$ is such that for each $g \in$ Lip 1, we have

$$(3.4.1) \qquad |L_n(g;x) - g(x)| \leq C_1 \frac{\sqrt{1 - x^2}}{n} \text{ for } -1 \leq x \leq 1.$$

Then for each $f \in C[-1,1]$, we have

$$(3.4.2) \qquad |L_n(f;x) - f(x)| \leq C_2 \omega\left(f; \frac{\sqrt{1 - x^2}}{n}\right) \text{ for } -1 \leq x \leq 1,$$

where C_2 is independent of f, x, and n.

The preceding is a specific case of the more general theorem stated by DeVore.

We next make the following observation which follows from the uniqueness of the $r_{k,n}(x)$'s.

$$(3.4.3) \qquad \sum_{k=0}^{n} r_{k,n}(x) \equiv 1$$

Proof of Theorem 3.1

Let $f \in \mathrm{Lip}\ 1$. Then using (3.4.3) we have

$$|R_n(f;x) - f(x)| = \left|\sum_{k=0}^{n} f(t_{k,n})\, r_{k,n}(x) - f(x)\right|$$

$$(3.4.4) \qquad = \left|\sum_{k=0}^{n} [f(t_{k,n}) - f(x)]\, r_{k,n}(x)\right|$$

$$\le \sum_{k=0}^{n} \lambda\, |x - t_{k,n}|\, |r_{k,n}(x)|$$

for some absolute constant λ.

Hence, using Lemma 3.10,

$$|R_n(f;x) - f(x)| = O\left(\sum_{k=0}^{n} |x - t_{k,n}|\, |r_{k,n}(x)|\right)$$

$$(3.4.5) \qquad = O\left(\frac{\sqrt{1 - x^2}}{n}\right) \text{ for } -1 \le x \le 1 .$$

From Lemma 3.11, the operator R_n is bounded. Hence, the theorem follows from Theorem 3.A.

CHAPTER FOUR
LACUNARY INTERPOLATION BY SPLINES

Introduction and Main Results

In the last two chapters, we considered Birkhoff interpolation problems using polynomials. The object of this chapter is to consider analogous problems using spline interpolation instead of polynomial interpolation. We shall limit ourselves to problems in the (0,1,3) and (0,1,2,4) cases. Before presenting our main results, we introduce some notation.

Denote by $S_{n,q}^{(r)}$ the class of splines S(x) such that

1) $S(x) \in C^r [a, b]$

2) S(x) is a polynomial of degree q in $[x_i, x_{i+1}]$; $i = 0, 1, \ldots, n - 1$.

Throughout this chapter, $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ will represent the joints of a spline. Also $h_i = x_{i+1} - x_i$, $2z_i = x_i + x_{i+1}$, and $\delta = \max\limits_{i=0,1,\ldots,n-1} (h_i)$..

The following are our main results.

Theorem 4.1

Given arbitrary numbers $f^{(j)}(x_i)$, $i = 0, 1, \ldots, n$; $j = 0, 1, 2$; $f^{(iv)}(z_i)$, $i = 0, 1, \ldots, n - 1$ where

$2z_i = x_i + x_{i+1}$, $f'''(x_o)$, $f'''(x_n)$ there exists a unique $S_n(x) \in S^{(4)}_{n,8}$ such that

$$(4.1.1) \qquad \begin{cases} S_n^{(j)}(x_i) = f^{(j)}(x_i), \; i = 0, 1, \ldots, n; \; j = 0, 1, 2 \\ S_n^{(iv)}(z_i) = f^{(iv)}(z_i), \; i = 0, 1, \ldots, n-1 \\ S_n'''(x_0) = f'''(x_0), \; S_n'''(x_n) = f'''(x_n) \end{cases}$$

Theorem 4.2

Let $f \in C^l\,[0,1]$, $l \geq 4$. Then, for the unique spline $S_n(x)$ associated with f and satisfying (4.1.1) we have

$$(4.1.2) \qquad |S_n^{(r)}(x) - f^{(r)}(x)| \leq c_{r,l}\delta^{l-r}\,\omega(f^{(l)}, \delta),$$

$r = 0,1,2,3,4$, $l = 4,5,6,7,8$. Also, for $f \in C^9\,[0,1]$

$$(4.1.3) \qquad \begin{aligned} &|S_n^{(r)}(x) - f^{(r)}(x)| \\ &\leq c_{r,9}\delta^{9-r} \max_{0 \leq x \leq 1} |f^{(9)}(x)|, \\ &r = 0, 1, 2, 3, 4. \end{aligned}$$

Theorem 4.3

Given arbitrary numbers $f(z_i)$ for $i = 0, 1, \ldots, n-1$; $f^{(j)}(x_i)$ for $i = 0, 1, \ldots, n$ where $j = 1, 2, 4$; and $f(x_0)$, $f(x_n)$; there exists a unique spline $T_n(x) \in S^{(4)}_{n,8}$ such that

$$(4.1.4)\quad \begin{cases} T_n(z_i) = f(z_i);\ i = 0, 1, \ldots, n-1 \\ T_n^{(j)} = f^{(j)}(x_i);\ i = 0, 1, \ldots, n;\ j = 1, 2, 4 \\ T_n(x_0) = f(x_0),\ T_n(x_n) = f(x_n) \end{cases}$$

Theorem 4.4

Let $f \in C^1[0,1]$. Then for the unique 8th degree spline $T_n(x)$ associated with f and satisfying (1.4), we have

$$(4.1.5)\quad |T_n^{(r)}(x) - f^{(r)}(x)| \le \alpha_{r,l}\, \delta^{l-r}\, \omega(f^{(l)}, \delta);$$

$r = 0, 1, 2, 3, 4;\ l = 4, 5, 6, 7, 8$

Also, for $f \in C^9\,[0,1]$,

$$(4.1.6)\quad |T_n^{(r)}(x) - f^{(r)}(x)| \le \alpha_{r,9}\, \delta^{9-r} \max_{0 \le x \le 1} |f^{(9)}(x)|;$$

$r = 0, 1, 2, 3, 4$

Next, we present our results for the (0,1,3) case.

Theorem 4.5

Given arbitrary numbers $f^{(j)}(x_i)$ for $i = 0, 1, \ldots, n$ where $j = 0,3$; $f^{(j)}(z_i)$ for $i = 0,1, \ldots, n-1$ where $j = 0,1$; and $f'(x_0)$, $f'(x_n)$; there exists a unique spline $R_n(x) \in S_{n,7}^{(3)}$ such that

$$(4.1.7)\quad \begin{aligned} R_n^{(j)}(x_i) &= f^{(j)}(x_i),\ i = 0, 1, \ldots, n;\ j = 0,3 \\ R_n^{(j)}(z_i) &= f^{(j)}(z_i);\ i = 0, 1, \ldots, n-1;\ j = 0,1 \\ R_n'(x_0) &= f'(x_0),\ R_n'(x_n) = f'(x_n) \end{aligned}$$

<u>Theorem 4.6</u>

Let $f \in C^1[0,1]$. Then, for the unique 7th degree spline associated with f and satisfying (4.1.7), we have;

$$(4.1.8) \qquad |R_n^{(r)}(x) - f^{(r)}(x)| \le \beta_{r,l}\delta^{l-r}\omega(f^{(l)}, \delta)$$

$r = 0, 1, 2, 3$; $l = 3, 4, 5, 6, 7$

Also, for $f \in C^8[0,1]$,

$$(4.1.9) \qquad |R_n^{(r)}(x) - f^{(r)}(x)| \le \beta_{r,8}\,\delta^{8-r} \max_{0\le x\le 1} |f^{(8)}(x)|;$$

$r = 0, 1, 2, 3$

Before continuing with the proofs of Theorems 4.1 - 4.6, we make some remarks. The splines discussed in Theorems 4.1 - 4.6 involve the use of midpoint data. It would be natural to ask why not interpolate data only at the joints. The preceding question is referred to in a remark by Demko [9] that, as in (0,2) interpolation by splines, the (0,1,2,4) and (0,1,3) cases are ill-poised for splines that interpolate data only at the joints.

By allowing the additional freedom of prescribing some data at the midpoints of the joints, we are able to obtain interpolating splines with the following positive properties.

1) The splines given in Theorems 4.1, 4.3, and 4.5 exist uniquely for any choice of knots.
2) As shown in the proofs of Theorems 4.1, 4.3, and 4.5, our splines are uniquely determined by a tridiagonal dominant system. In Theorem 4.1, the

spline is actually determined by a diagonal system. This spline, consequently, is a locally determined process.

3) Unlike many G-splines and piecewise polynomial schemes, our splines are "very" smooth. The (0,1,2,4) splines are of the class C^4 [0,1], while the (0,1,3) spline is of the class C^3 [0,1].

4) Theorems 4.2, 4.4, and 4.6 show that our splines converge with the optimal order of convergence for splines of their respective classes.

Proofs of Theorems

In this section, we present proofs of the Theorems 4.1 - 4.6.

Proof of Theorem 4.1

If Q(z) is any polynomial of degree eight on [0, 1], then we have

$$\begin{aligned} Q(z) &= Q(0)A_0(z) + Q(1)A_0(1-z) + Q'(0)B_0(z) \\ &\quad - Q'(1)B_0(1-z) + Q''(0)C_0(z) + Q''(1)C_0(1-z) \\ &\quad + Q'''(0)D_0(z) - Q'''(1)D_0(1-z) + Q^{(iv)}(\tfrac{1}{2})\, E_1(z), \end{aligned} \tag{4.2.1}$$

where

$$A_0(z) = 1 - 35z^4 + 84z^5 - 70z^6 + 20z^7,$$

$$B_0(z) = z - \frac{70}{3}z^4 + \frac{175}{3}z^5 - 56z^6 + \frac{70}{3}z^7 + \frac{10}{3}z^8,$$

$$C_o(z) = \frac{z^2}{2} - \frac{20}{3}z^4 + \frac{50}{3}z^5 - \frac{35}{2}z^6 + \frac{26}{3}z^7 - \frac{5}{3}z^8,$$

$$D_o(z) = \frac{z^3}{6} - \frac{5}{6}z^4 + \frac{5}{3}z^5 - \frac{5}{3}z^6 + \frac{5}{6}z^7 - \frac{z^8}{6},$$

and

$$E_1(z) = \frac{z^4}{9} - \frac{4}{9}z^5 + \frac{2}{3}z^6 - \frac{4}{9}z^7 + \frac{z^8}{9}.$$

Next, we let $x = x_i + th_i$, $h_i = x_{i+1} - x_i$, $0 \le t \le 1$. Here, our object is to prove that there exists a unique $S_n(x) \in S_{n,8}^{(4)}$ satisfying the conditions of Theorem 4.1. We define $S_n(x)$ by the following representation.

$$\begin{aligned} S_n(x) &= f_i A_0(t) + f_{i+1} A_0(1-t) \\ &+ h_i f_i' B_0(t) - h_i f_{i+1}' B_0(1-t) \\ &+ h_i^2 f_i'' C_0(t) + h_i^2 f_{i+1}'' C_0(1-t) \\ &+ h_i^3 S_n'''(x_i) D_0(t) \\ &- h_i^3 S_n'''(x_{i+1}) D_0(1-t) + h_i^4 f_{i+\frac{1}{2}}^{(iv)} E_1(t) \end{aligned} \tag{4.2.2}$$

where

$$f_i^{(p)} = f^{(p)}(x_i), \quad f_{i+1}^{(p)} = f^{(p)}(x_{i+1}), \quad f_{i+\frac{1}{2}}^{(iv)} = f^{(iv)}(z_i).$$

Next, we set

$$S_n'''(0) = f'''(0), \quad S_n'''(1) = f'''(1). \tag{4.2.3}$$

Clearly, $S_n(x)$ as defined by (4.2.2) belongs to C^3 [0,1], no matter how we choose $S_n'''(x_i)$, $i = 1, 2, \ldots, n-1$. They are uniquely determined by the conditions

$$S_n^{(iv)}(x_i +) = S_n^{(iv)}(x_i -),\ i = 1, 2, \ldots, n-1. \tag{4.2.4}$$

A simple computation shows that (4.2.4) gives

$$\begin{aligned} 5h_i^3 h_{i-1}^3 (h_i + h_{i-1})\ S_n'''(x_i) &= 210\,(h_{i-1}^4 f_{i+1} - h_i^4 f_{i-1} \\ &\quad + (h_i^4 - h_{i-1}^4)\, f_i) - 70\, h_i h_{i-1}\, (h_{i-1}^3 f_{i+1}' + h_i^3 f_{i-1}' \\ &\quad + 2\,(h_i^3 + h_{i-1}^3)\, f_i') + 5h_i^2 h_{i-1}^2 (h_{i-1}^2 f_{i+1}''\ h_i^2 f_{i-1}'' \\ &\quad + 8\,(h_i^2 - h_{i-1}^2)\, f_i'') + \frac{2}{3} h_i^4 h_{i-1}^4 \left(f_{i+\frac{1}{2}}^{(iv)} - f_{i-\frac{1}{2}}^{(iv)}\right). \end{aligned} \tag{4.2.5}$$

Thus, $S_n(x)$ defined by (4.2.2), (4.2.3) and (4.2.5) satisfy all the conditions of the theorem. This completes the proof of Theorem 4.1.

<u>Proof of Theorem 4.2</u>

To begin, the following identities are valid:

(4.2.6)

$$\begin{cases} A_0(t) + A_0(1-t) = 1,\ A_0(1-t) + B_0(t) - B_0(1-t) = t, \\ A_0(1-t) - 2B_0(1-t) + 2C_0(t) + C_0(1-t) = t^2, \\ A_0(1-t) - 3B_0(1-t) + 6C_0(1-t) + 6D_0(t) - 6D_0(1-t) = t^3 \\ A_0(1-t) - 4B_0(1-t) + 12C_0(1-t) - 24D_0(1-t) + 24E_1(t) = t^4 \\ A_0(1-t) - 5B_0(1-t) + 20C_0(1-t) - 60D_0(1-t) + 60E_1(t) = t^5 \\ A_0(1-t) - 6B_0(1-t) + 30C_0(1-t) - 120D_0(1-t) + 90E_1(t) = t^6 \\ A_0(1-t) - 7B_0(1-t) + 42c_0(1-t) - 210D_0(1-t) + 105E_1(t) = t^7 \\ A_0(1-t) - 8B_0(1-t) + 56C_0(1-t) - 336D_0(1-t) + 105E_1(t) = t^8 \end{cases}$$

The identities can be derived from (4.2.4) and the uniqueness of this interpolation formula.

Next, from Taylor's formula we have

$$f^{(1)}(x_{i+1}) = \sum_{j=1}^{p-1} \frac{f^{(j)}(x_i)}{(j-1)!} h_i^{j-1} + \frac{f^{(p)}(\eta_{1,1}\, h_i^{p-1}}{(p-1)!}, \tag{4.2.7}$$

$$\begin{aligned} f^{(1)}(x_{i-1}) &= \sum_{j=1}^{p-1} \frac{f^{(j)}(x_i)}{(j-1)!} (-h_{i-1})^{j-1} \\ &+ \frac{f^{(p)}(\eta_{2,1})(-h_{i-1})^{p-1}}{(p-1)!} \end{aligned} \tag{4.2.8}$$

$$\begin{aligned} f^{(iv)}(z_i) &= \sum_{j=4}^{p-1} \frac{f^{(j)}(x_i)}{(j-4)!} \left(\frac{h_i}{2}\right)^{j-4} \\ &+ \frac{f^{(p)}(\eta_3)\left(\frac{h_i}{2}\right)^{p-4}}{(p-4)!}, \end{aligned} \tag{4.2.9}$$

and

$$\begin{aligned} f^{(iv)}(z_i - 1) &= \sum_{j=4}^{p-1} \frac{f^{(j)}(x_i)\left(-\frac{h_{i-1}}{2}\right)^{j-4}}{(j-4)!} \\ &+ \frac{f^{(p)}(\eta_4)\left(-\frac{h_{i-1}}{2}\right)^{p-4}}{(p-4)!}, \end{aligned} \tag{4.2.10}$$

where $x_i < \eta_{1,1} < x_{x+1}$, $x_{x-1} < \eta_{2,1} < x_i$, $x_i < \eta_3 < z_i$, and $z_{i-1} < \eta_4 < x_i$.

Let $t = \dfrac{x-x_i}{h_i}$. Then

$$f(x) = f_i + (x-x_i)\, f_i' + \frac{(x-x_i)^2}{2!}\, f_i'' + \frac{(x-x_i)^3}{3!}\, f_i'''$$

$$(4.2.11) \qquad + \frac{(x-x_i)^4}{4!}\, f^{(iv)}(\eta_0) = f_i + th_i f_i' + \frac{(th_i)^2}{2!} f_i''$$

$$+ \frac{(th_i)^3}{3!}\, f_i''' + \frac{(th_i)^4}{4!}\, f^{(iv)}\,(\eta_0)\,.$$

Let $f \in C^p\,[0,1]$, $4 \le p \le 8$, then from (4.2.5) and (4.2.7) - (4.2.10),

$$|f'''(x_i) - S_n'''(x_i)| \le \frac{c_p h_i h_{i-1}\left(h_i^{p-4} + h_{i-1}^{p-4}\right)}{h_i + h_{i-1}}\, \omega\,(f^{(p)},\, \delta) \text{ and}$$

for $f \in C^9\,[0,1]$

$$(4.2.12) \qquad \begin{aligned} &|f'''(x_i) - S_n'''(x_i)| \\ &\le \frac{c_p h_i h_{i-1}\left(h_i^5 + h_{i-1}^5\right)}{h_i + h_{i-1}} \max_{0 \le x \le 1} |f^{(9)}(x)|, \end{aligned}$$

where $\delta = \max\limits_{i=0,1,\ldots,n-1} (h_i)$ and $\omega\,(f,\delta)$ denotes the modulus of continuity of f.

Next, we let $x = x_i + th_i$, $h_i = x_{i+1} - x_i$ and $f \in C^4\,[0,1]$. Then using (4.2.2), we obtain

$$(4.2.13) \qquad S_n(x) - f(x) = \lambda_i(t) + \mu_i(t)$$

where

$$\lambda_i(t) = h_i^3 (S_n'''(x_i) - f'''(x_i))\, D_0(t) - h_i^3\, (S_n'''(x_{i+1}) \tag{4.2.14}$$
$$- f'''(x_{i+1}))\, D_0(1 - t)$$

and

$$\mu_i(t) = f_i A_0(t) + f_{i+1} A_0(1 - t) + h_i (f_i' B_0(t)$$
$$- f_{i+1}' B_0(1 - t)) + h_i^2 (f_i'' C_0(t) + f_{i+1}'' C_0(1 - t)) \tag{4.2.15}$$
$$+ h_i^3 (f_i''' D_0(t) - f_{i+1}''' D_0(1 - t)) + h_i^4 f_{i+\frac{1}{2}}^{(iv)} E_1(t) - f(x).$$

Next, on using (4.2.12) and (4.2.14) we obtain

$$|\lambda_i(t)| \le C_1 h_i^4 \max_{0 \le t \le 1} |D_0(t)|\, \omega(f^{(4)}, \delta). \tag{4.2.16}$$

Also, from (4.2.6), (4.2.7), (4.2.11), and (4.2.15), we have

$$\mu_i(t) = f_i A_0(t) + A_0(1 - t) \left[\sum_{j=0}^{3} \frac{f_i^{(j)} h_i^j}{j!} + \frac{f^{(iv)}(\eta_{1,0})}{4!} h_i^4\right]$$
$$+ h_i f_i' B_0(t) - h_i \left[\sum_{j=1}^{3} \frac{f_i^{(j)} h_i^{j-1}}{(j - 1)!} + \frac{f^{(iv)}(\eta_{1,1})}{3!} h_i^3\right] B_0(1 - t)$$
$$+ h_i^2 f_i'' C_0(t) + h_i^3 f_i''' D_0(t) + h_i^4 f_{i+\frac{1}{2}}^{(iv)} E_1(t)$$
$$+ h_i^2 C_0(1 - t) \left[\sum_{j=2}^{3} \frac{f_i^{(j)}}{(j - 2)!} h_i^{j-2} + \frac{f^{(iv)}(\eta_{1,2})}{2!} h_i^2\right]$$
$$- h_i^3 \left[f_i''' + h_i f^{(iv)}(\eta_{1,3})\right] D_0(1 - t) \tag{4.2.17}$$
$$- \left(f_i + t h_i f_i' + \frac{t^2 h_i^2}{2!} f_i'' + \frac{t^3 h_i^3}{3!} f_i''' + \frac{t^4 h_i^4 f^{(iv)}(\eta_0)}{4!}\right)$$
$$= h_i^4 \left[\frac{f^{(iv)}(\eta_{0,1})}{24} A_0(1 - t) - \frac{f^{(iv)}(\eta_{1,1})}{6} B_0(1 - t)\right.$$
$$+ \frac{f^{(iv)}(\eta_{1,2})}{2} C_0(1 - t) - f^{(iv)}(\eta_{1,3})\, D_0(1 - t)$$
$$\left. + f^{(iv)}(z_i)\, E_1(t) - \frac{t^4}{24} f^{(iv)}(\eta_0)\right].$$

On using the identity given in (4.2.6) corresponding to t^4, we have

$$\begin{aligned}\mu_i(t) = h_i^4 \Bigg[&\left(\frac{f^{(iv)}(\eta_{1,0}) - f^{(iv)}(\eta_0)}{24}\right) A_0(1-t) \\ &+ \left(\frac{f^{(iv)}(\eta_0) - f^{(iv)}(\eta_{1,1})}{6}\right) B_0(1-t) \\ &+ \left(\frac{f^{(iv)}(\eta_{1,2}) - f^{(iv)}(\eta_0)}{2}\right) C_0(1-t) \\ &+ \left(f^{(iv)}(\eta_0) - f^{(iv)}(\eta_{1,3})\right) D_0(1-t) \\ &+ \left(f^{(iv)}(z_i) - f^{(iv)}(\eta_0)\right) E_1(t)\Bigg].\end{aligned} \tag{4.2.18}$$

From this we obtain

$$|\mu_i(t)| \le C_6 h_i^4\, \omega(f^{(iv)}, \delta) \tag{4.2.19}$$

On combining (4.2.13), (4.2.16), and (4.2.19), we obtain (4.1.2) for r = 0, 1 = 4. Proof in the other cases are similar. We omit the details.

<u>Proof of Theorem 4.3</u>

We begin by making the following observations: If Q(z) is any polynomial of degree eight on [0,1], then we have

$$\begin{aligned}Q(t) &= Q(0)A_0(t) + Q(1)A_0(1-t) + Q(\tfrac{1}{2})A_1(t) \\ &\quad Q'(0)B_0(t) - Q'(1)B_0(1-t) + Q''(0)C_0(t) \\ &\quad Q''(1)C_0(1-t) + Q^{(4)}(0)E_0(t) + Q^{(4)}(1)E_0(1-t)\end{aligned} \tag{4.2.20}$$

where

$$7A_0(t) = -384t^8 + 1578t^7 - 2323t^6 + 1299t^5 - 177t^3 + 7$$

$$7A_1(t) = 768t^8 - 3072t^7 + 4352t^6 - 2304t^5 + 256t^3$$

$$7B_0(t) = -122t^8 + 509t^7 - 766t^6 + 443t^5 - 71t^3 + 7t$$

$$14C_0(t) = -26t^8 + 111t^7 - 173t^6 + 106t^5 - 25t^3 + 7t^2$$

$$840E_0(t) = 10t^8 - 47t^7 + 87t^6 - 79t^5 + 35t^4 - 6t^3$$

Let $x \in [x_i, x_{i+1}]$. Then $x = x_i + th_i$ where $0 \le t \le 1$. Our goal is to prove that there exists a unique spline $T_n(x) \in S_{n,8}^{(4)}$ satisfying (4.1.4). We define $T_n(x)$, for $x_i \le x \le x_{i+1}$, by

$$\begin{aligned} T_n(x) &= T_n(x_i)A_0(t) + T_n(x_{i+1})A_0(1-t) + f(z_i)A_1(t) \\ &+ h_i f'(x_i)B_0(t) - h_i f'(x_{i+1})B_0(1-t) + h_i^2 f''(x_i)C_0(t) \\ &+ h_i^2 f''(x_{i+1})C_0(1-t) + h_i^4 f^{(4)}(x_i)E_0(t) \\ &+ h_i^4 f^{(4)}(x_{i+1})E_0(1-t) \end{aligned} \tag{4.2.21}$$

Next, we set $T_n(x_0) = f(x_0)$, $T_n(x_n) = f(x_n)$. Clearly, $T_n(x)$ is continuous on [0,1] no matter how we define $T_n(x_i)$; $i = 1, \ldots, n-1$. We shall uniquely determine the unknowns $T_n(x_i)$ by the conditions $T_n^{(3)}(x_i+) = T_n^{(3)}(x_i-)$; $i = 1, \ldots, n-1$. A simple computation gives, for $i = 1, 2, \ldots, n-1$,

$$T_n(x_{i-1})\left[474h_i^3\right] + T_n(x_i)\left[1062h_i^3 + 1062h_{i-1}^3\right] + T_n(x_{i+1})\left[474h_{i-1}^3\right]$$

$$= f(z_{i-1})\left[1536h_i^3\right] + f(z_i)\left[1536h_{i-1}^3\right]$$

$$+ f'(x_{i-1})\left[-132h_{i-1}h_i^3\right] + f'(x_i)\left[426h_{i-1}h_i^3 - 426h_{i-1}^3h_i\right]$$

$$(4.2.22) \quad + f'(x_{i+1})\left[132h_{i-1}^3h_i\right] + f''(x_{i-1})\left[-12h_{i-1}^2h_i^3\right]$$

$$+ f''(x_i)\left[-75h_{i-1}^2h_i^3 - 75h_{i-1}^3h_i^2\right] + f''(x_{i+1})\left[-12h_{i-1}^3h_i^2\right]$$

$$+ f^{(4)}(x_{i-1})\left[\frac{1}{20}h_{i-1}^4h_i^3\right] + f^{(4)}(x_i)\left[-\frac{3}{10}h_{i-1}^4h_i^3 - \frac{3}{10}h_{i-1}^3h_i^4\right]$$

$$+ f^{(4)}(x_{i+1})\left[\frac{1}{20}h_{i-1}^3h_i^4\right].$$

Note that the equations in (4.2.22) make up a tridiagonal dominant system. Such a system has a unique solution. The unique spline $T_n(x)$ satisfying (4.2.21) and (4.2.22) provides the proof for Theorem 4.3.

Proof of Theorem 4.4

The technique used to prove Theorem 4.2 can be applied here with only minor changes. Again, we define, for $x = x_i + th_i$, $h_i = x_{i+1} - x_i$,

$$(4.2.23) \quad T_n(x) - f(x) = \lambda_i(t) + \mu_i(t)$$

where

$$(4.2.24) \quad \lambda_i(t) = (T_n(x_i) - f(x_i))A_0(t) + (T_n(x_{i+1}) - f(x_{i+1}))A_0(1-t)$$

and

$$\mu_i(t) = f_iA_0(t) + f_{i+1}A_0(1-t) + f_{i+\frac{1}{2}}A_1(t)$$

$$(4.2.25) \quad + h_i(f_i'B_0(t) - f_{i+1}'B_0(1-t)) + h_i^2\left(f_i''C_0(t) + f_{i+1}''C_0(1-t)\right)$$

$$+ h_i^4\left(f_i^{(4)}E_0(t) + f_{i+1}^{(4)}E_0(1-t)\right) - f(x).$$

We may estimate $\mu_1(t)$ in exactly the same way as in the proof of Theorem 4.2. The estimate of $\lambda_1(t)$, however, is slightly different. After referring back to the tridiagonal dominant system in (4.2.22), we observe that

$$\begin{aligned} &\left|[T_n(x_{i-1}) - f(x_{i-1})]\,474h_i^3 + [T_n(x_i) - f(x_i)]\left[1062h_i^3 + 1062h_{i-1}^3\right]\right. \\ (4.2.26)\qquad &\left. + [T_n(x_{i+1}) - f(x_{i+1})]\,474h_{i-1}^3\right| \\ &\le |T_n(x_i) - f(x_i)|\left[588h_i^3 + 588h_{i-1}^3\right] \end{aligned}$$

when

$$|T_n(x_i) - f(x_i)| = \max_{j=i-1,i,i+1} |T_n(x_j) - f(x_j)|.$$

Now, we use (4.2.26) along with the technique from the proof of Theorem 4.2 to finish the estimate.

<u>Proof of Theorem 4.5</u>

The proof of Theorem 4.5 follows in exactly the same way as the proof of Theorem 4.3. We, therefore, only list the equations which provide the unique explicit form for the spline $R_n(x) \in S_{n,7}^{(3)}$ satisfying (4.1.7).

Let $x \in [x_i, x_{i+1}]$. Then $x = x_i + th_i$ where $0 \le t \le 1$.

$$\begin{aligned} &R_n(x) = f(x_i)A_0(t) + f(z_i)A_1(t) + f(x_{i+1})A_0(1-t) \\ (4.2.27)\qquad &+ h_iR_n'(x_i)B_0(t) + h_if'(z_i)B_1(t) - h_iR_n'(x_{i+1})B_0(1-t) \\ &+ h_i^3f'''(x_i)D_0(t) - h_i^3f'''(x_{i+1})D_0(1-t) \quad \text{for } x \in [x_i, x_{i+1}] \end{aligned}$$

where

$$
\begin{aligned}
&12A_0(t) = -600t^7 + 2228t^6 - 2946t^5 + 1475t^4 + 169t^2 + 12 \\
&3A_1(t) = -64t^6 + 192t^5 - 160t^4 + 32t^2 \\
(4.2.28) \quad &24B_0(t) = -216t^7 + 820t^6 - 1122t^5 + 595t^4 - 101t^2 + 24t \\
&B_1(t) = -32t^7 + 112t^6 - 136t^5 + 60t^4 - 4t^2 \\
&288D_0(t) = 24t^7 - 100t^6 + 162t^5 - 127t^4 + 48t^3 - 7t^2
\end{aligned}
$$

And the unknowns $T_n(x_i)$ for i = 1, ..., n - 1 are determined by the following tridiagonal dominant system of equations. For i = 1, 2, ..., n - 1, we have

$$
\begin{aligned}
&R_n'(x_{i-1})\left[13h_{i-1}h_i^2\right] + R_n'(x_i)\left[101h_{i-1}h_i^2 + 101h_{i-1}^2h_i\right] + R_n'(x_{i+1}\left[13h_{i-1}^2h_i\right] \\
&= f(x_{i-1})\left[-82h_i^2\right] + f(z_{i-1})\left[-256h_i^2\right] + f(x_i)\left[338h_i^2 - 338h_{i-1}^2\right] \\
(4.2.29) \quad &+ f(z_i)\left[256h_{i-1}^2\right] + f(x_{i+1})\left[82h_{i-1}^2\right] + f'(z_{i-1})\left[-96h_{i-1}h_i^2\right] \\
&+ f'(z_i)\left[-96h_{i-1}^2h_i\right] + f'''(x_{i-1})\left[\frac{1}{12}h_{i-1}^3h_i^2\right] \\
&+ f'''(x_i)\left[-\frac{7}{12}h_{i-1}^3h_i^2 - \frac{7}{12}h_{i-1}^2h_i^3\right] + f'''(x_{i+1})\left[\frac{1}{12}h_{i-1}^2h_i^3\right].
\end{aligned}
$$

<u>Proof of Theorem 4.6</u>.

Along the same lines as the proof of Theorem 4.4.

CHAPTER FIVE
EXTREMAL PROPERTIES FOR THE DERIVATIVES OF ALGEBRAIC POLYNOMIALS

Main Results

In this chapter, we present three theorems which discuss the growth of the derivatives of algebraic polynomials bounded by curved majorants in the L^p norm. Specifically, Theorems 5.1 and 5.2 state results in the L^2 norm. Note that Theorems 5.1 and 5.2 give the polynomials that possess derivatives with the maximum possible value for every n. Theorem 5.3 presents polynomials that possess derivatives of maximum value asymptotically. We now present our results.

Throughout this chapter, $T_n(x)$ and $u_{n-1}(x)$ are defined by $T_n(x) = \cos n\theta$ and $u_{n-1}(x) = \dfrac{\sin n\theta}{\sin\theta}$ where $x = \cos\theta$.

Theorem 5.1

Let $p_{n+1}(x)$ be a real algebraic polynomial of degree $n + 1$ such that $|p_{n+1}(x)| \le (1 - x^2)^{\frac{1}{2}}$, for $-1 \le x \le 1$.

Then, for k = 2, 3, ...; we have:

$$\int_{-1}^{1} \left[p_{n+1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-3}{2}}\, dx \qquad (5.1.1)$$

$$\le \int_{-1}^{1} \left[f_o^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-3}{2}}\, dx$$

where $f_o(x) = (1 - x^2)\, u_{n-1}(x)$. Equality iff $p_{n+1}(x) = \pm f_o(x)$.

Remark 1.

The case k = 1, under the further assumption that the polynomial $p_{n+1}(x)$ has all real zeros that lie inside [-1,1], is also treated in [44].

Next, we shall prove:

Theorems 5.2

Let $p_{n+2}(x)$ be a real algebraic polynomial of degree n + 2 such that $|p_{n+2}(x)| \le 1 - x^2$, for $-1 \le x \le 1$.

Then for k = 3, 4, ...; we have

$$\begin{aligned}\int_{-1}^{1} \left[p_{n+2}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-5}{2}}\, dx \\ \le \int_{-1}^{1} \left[f_1^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-5}{2}}\, dx\end{aligned} \tag{5.1.2}$$

where $f_1(x) = (1 - x^2)\, T_n(x)$. Equality iff $p_{n+2}(x) = \pm f_1(x)$.

Remark 2:

In the case k = 2, we were not able to resolve the inequality (5.1.2).

Theorem 5.3

Let $P_{n+2}(x)$ be an arbitrary polynomial of degree n + 2 such that $|P_{n+2}(x)| \le 1 - x^2$ for $-1 \le x \le 1$. Then

$$\int_{-1}^{1} |P_{n+2}'(x)|^{2p}\, dx \le \frac{2}{2p + 1}\, n^{2p} + C_p n^{2p-1} \tag{5.1.3}$$

where p is any fixed positive integer and C_p is a constant that depends on p. The preceding is best possible in the following sense.

$$\lim_{n\to\infty} \frac{1}{n^{2p}} \int_{-1}^{1} |P'_{n+2}(x)|^{2p}\, dx = \frac{2}{2p+1},$$

where $P_{n+2}(x) = (1 - x^2)\, T_n(x)$.

Lemmas

Here we state and prove some lemmas which are needed in the proofs our theorems. We now state:

Lemma 5.1

Let $q_{n-1}(x)$ be any algebraic polynomial of degree at most n - 1 with real coefficients. Further, let

(5.2.1) $\quad |q_{n-1}(x)| \le (1 - x^2)^{-\frac{1}{2}}$, for $-1 < x < 1$.

Then we have

(5.2.2) $$\int_{-1}^{1} \left[q'_{n-1}(x)\right]^2 (1 - x^2)^{\frac{3}{2}}\, dx \le \frac{\pi}{2}(n^2 - 1)$$

Equality iff $q_{n-1}(x) = \pm \dfrac{\sin n\theta}{\sin\theta}$, $x = \cos\theta$. Proof of this lemma is given in [45].

Lemma 5.2

Let $q_{n-1}(x)$ be any algebraic polynomial of degree n - 1 with real coefficients such that $|q_{n-1}(x)| \le (1 - x^2)^{-1/2}$, for $-1 < x < 1$. Then we have for $k = 1, 2, \ldots ..$

$$(5.2.3)\qquad \int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\,dx \le \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\,dx$$

Equality iff $q_{n-1}(x) = \pm\, u_{n-1}(x)$; $u_{n-1}(x) = \dfrac{\sin n\theta}{\sin\theta}$, $x = \cos\theta$.

Proof

We begin by setting $q_{n-1}(x) = \sum_{j=0}^{n-1} \beta_j u_j(x)$.

Now using the orthogonal properties of $\{u_j^{(k)}(x)\}$ and $\{u_j'(x)\}$, we obtain

$$(5.2.4)\qquad \begin{aligned} &\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\,dx \\ &= \sum_{j=k}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}}\,dx \end{aligned}$$

and

$$(5.2.5)\qquad \int_{-1}^{1} \left[q_{n-1}'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\,dx = \sum_{j=1}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1-x^2)^{\frac{3}{2}}\,dx$$

Next, we note that $y = u_j(x)$ satisfies the differential equation

$$(5.2.6)\qquad (1-x^2)\,y'' - 3xy' + j(j+2)\,y = 0,$$

From (5.2.6) it follows that

$$(1 - x^2)\, u_j^{(k)}(x) - (2k - 1)\, x u_j^{(k-1)}(x) + ((j + 1)^2 - (k - 1)^2)\, u_j^{(k-2)}(x) = 0 \tag{5.2.7}$$

Now on using integration by parts and (5.2.7), we have

$$\begin{aligned}
&\int_{-1}^{1} \left[u_j^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k+1}{2}} dx \\
&= -\int_{-1}^{1} \left[u_j^{(k)}(x)\right](1 - x^2)^{\frac{2k-1}{2}} \\
&\cdot \left[(1 - x^2)\, u_j^{(k+1)}(x) - (2k + 1)\, x u_j^{(k)}(x)\right] dx \\
&= \left(\int_{-1}^{1} \left[u_j^{(k-1)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx\right) \left[(j + 1)^2 - k^2\right]
\end{aligned} \tag{5.2.8}$$

Through repeated application of (5.2.8), we have

$$\begin{aligned}
&\int_{-1}^{1} \left[u_j^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k+1}{2}} dx \\
&= \left[\prod_{i=2}^{k} ((j + 1)^2 - i^2)\right] \int_{-1}^{1} \left[u_j'(x)\right]^2 (1 - x^2)^{\frac{3}{2}} dx
\end{aligned} \tag{5.2.9}$$

for k = 1, 2, ... where we define for k = 1,

$\prod_{i=2}^{k} ((j + 1)^2 - i^2) \equiv 1.$

Hence,

$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k+1}{2}}$$

$$= \sum_{j=k}^{n-1} \left[\prod_{i=2}^{k} ((j + 1)^2 - i^2)\right] \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1 - x^2)^{\frac{3}{2}}\, dx$$

$$\le \left[\prod_{i=2}^{k} (n^2 - i^2)\right] \sum_{j=1}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1 - x^2)^{\frac{3}{2}}\, dx$$

$$\le \left[\prod_{i=2}^{k} (n^2 - i^2)\right] \sum_{j=1}^{n-1} \beta_j^2 \int_{-1}^{1} \left[u_j'(x)\right]^2 (1 - x^2)^{\frac{3}{2}}\, dx$$

$$= \left[\prod_{i=2}^{k} (n^2 - i^2)\right] \int_{-1}^{1} \left[q_{n-1}'(x)\right]^2 (1 - x^2)^{\frac{3}{2}}\, dx$$

$$\le \left[\prod_{i=2}^{k} (n^2 - i^2)\right] \int_{-1}^{1} \left[u_{n-1}'(x)\right]^2 (1 - x^2)^{\frac{3}{2}}\, dx$$

$$= \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k+1}{2}} dx$$

Equality, iff $q_{n-1}(x) = \pm\, u_{n-1}(x)$.

This completes the proof of Lemma 5.2.

Lemma 5.3

Let $q_{n-1}(x)$ be any real algebraic polynomial of degree $n - 1$ such that $|q_{n-1}(x)| \le (1 - x^2)^{-\frac{1}{2}}$, for $-1 < x < 1$.

Then, we have for $k = 0, 1, \ldots$

$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx \le \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx \tag{5.2.10}$$

Equality holds iff $q_{n-1}(x) = \pm\, u_{n-1}(x)$.

Proof

Let $v_1, v_2, \ldots, v_{n-k}$ be the zeros of $T_n^{(k)}(x)$. Then

(5.2.11) $\quad |q_{n-1}^{(k)}(v_i)| \le |u_{n-1}^{(k)}(v_i)|$ for $i = 1, 2, \ldots, n-k$

For the proof of the above statement we refer to [45].

Now, using Gaussian quadrature formula, based on $v_1, v_2, \ldots, v_{n-k}$, we obtain

$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx = \sum_{i=1}^{n-k} \left[q_{n-1}^{(k)}(v_i)\right]^2 M_i$$

where $M_i = \displaystyle\int_{-1}^{1} \frac{\left[T_n^{(k)}(x)\right]^2}{(x-v_i)^2 \left[T_n^{(k+1)}(v_i)\right]^2} (1-x^2)^{\frac{2k-1}{2}} dx \ge 0$

for $i = 1, 2, \ldots, n-k$. In view of (5.2.11), we have

$$\int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx \le \sum_{i=1}^{n-k} \left[u_{n-1}^{(k)}(v_i)\right]^2 M_i$$

$$= \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx$$

This completes the proof of Lemma 5.3.

The previous three lemmas are needed for the proof of Theorem 5.1, while the subsequent three lemmas are used in the proof of Theorem 5.2.

Lemma 5.4

Let $q_{n-1}(x)$ be any real algebraic polynomial of degree at most $n-1$ such that $|q_{n-1}(x)| \le 1$ for $-1 \le x \le 1$.

Then we have:

(5.2.12) $$\int_{-1}^{1} \left[q_{n-1}'(x)\right]^2 (1 - x^2)^{\frac{1}{2}}\, dx \le \frac{\pi}{2}(n - 1)^2,$$

Equality iff $q_{n-1}(x) = \pm \cos(n - 1)\theta$, $x = \cos\theta$.

Proof of this lemma follows from a known result of Calderon and Klein [8].

Lemma 5.5

Let $q_n(x)$ by any real algebraic polynomial of degree n such that $|q_n(x)| \le 1$, for $-1 \le x \le 1$. Then, we have for $k = 1, 2, \ldots$

(5.2.13) $$\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx \le \int_{-1}^{1} \left[T_n^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx$$

Equality iff $q_n(x) = \pm T_n(x)$; $T_n(x) = \cos n\theta$, $x = \cos\theta$.

Proof

Let $q_{n'}(x) = \sum_{j=0}^{n} \alpha_j T_j(x)$. From the orthogonal properties of $\{T_j^{(k)}(x)\}$ and $\{T_j'(x)\}$, we obtain

(5.2.14) $$\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx = \sum_{j=k}^{n} \alpha_j^2 \int_{-1}^{1} \left[T_j^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx$$

and

(5.2.15) $$\int_{-1}^{1} \left[q_n'(x)\right]^2 (1 - x^2)^{\frac{1}{2}} dx = \sum_{j=1}^{n} \alpha_j^2 \int_{-1}^{1} \left[T_j'(x)\right]^2 (1 - x^2)^{\frac{1}{2}} dx$$

Following as in (5.2.8), we obtain

$$(5.2.16)\quad \int_{-1}^{1}\left[T_j^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx = \left[\prod_{i=1}^{k-1}(j^2-i^2)\right]\int_{-1}^{1}\left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

where for k = 1, we define $\prod_{i=1}^{k-1}(j^2-i^2) \equiv 1$.

Hence,

$$\int_{-1}^{1}\left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx$$

$$= \sum_{j=k}^{n}\left[\prod_{i=1}^{k-1}(j^2-i^2)\right]\alpha_j^2\int_{-1}^{1}\left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

$$\le \prod_{i=1}^{k-1}(n^2-i^2)\sum_{j=k}^{n}\alpha_j^2\int_{-1}^{1}\left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

$$\le \prod_{i=1}^{k-1}(n^2-i^2)\sum_{j=1}^{n}\alpha_j^2\int_{-1}^{1}\left[T_j'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

$$= \prod_{i=1}^{k-1}(n^2-i^2)\int_{-1}^{1}\left[q_n'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

$$\le \prod_{i=1}^{k-1}(n^2-i^2)\int_{-1}^{1}\left[T_n'(x)\right]^2 (1-x^2)^{\frac{1}{2}}\,dx$$

$$= \int_{-1}^{1}\left[T_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}\,dx$$

This completes the proof of Lemma 5.5.

Lemma 5.6

Let $q_n(x)$ be any real algebraic polynomial of degree n such that $|q_n(x)| \le 1$, for $-1 \le x \le 1$.

Then, we have for k = 1, 2, ...

(5.2.17) $$\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx \le \int_{-1}^{1} \left[T_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

Equality holds iff $q_n(x) = \pm T_n(x)$.

<u>Proof</u>

Let $u_1, u_2, \ldots, u_{n-k+1}$ be the zeros of $T_n^{(k-1)}(x)$.

Then

(5.2.18) $\left|q_n^{(k)}(u_i)\right| \le \left|T_n^{(k)}(u_i)\right|$, for $i = 1, 2, \ldots, n-k+1$

Equality possible for any i if $q_n(x) = \pm T_n(x)$.

For the proof of the above statement we refer to [45], page 104, formula (2.7.1). Now, using Gaussian quadrature formula, based on $u_1, u_2, \ldots, u_{n-k+1}$, we obtain

$$\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx = \sum_{i=1}^{n-k+1} \left[q_n^{(k)}(u_i)\right]^2 H_i$$

where

$$H_i = \int_{-1}^{1} \left[\frac{T_n^{(k-1)}(x)}{(x-u_i)\, T_n^{(k)}(u_i)}\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx \ge 0$$

Now, using (5.2.18), we have

$$\int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx \le \sum_{i=1}^{n-k+1} \left[T_n^{(k)}(u_i)\right]^2 H_i$$

$$= \int_{-1}^{1} \left[T_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

From this Lemma 5.6 follows.

The final lemma is used in the proof of Theorem 5.3.

Lemma 5.7

Let $t_n(\theta)$ be a trigonometric polynomial of order n. For p and k fixed positive integers, we have the following result.

(5.2.19) $$\int_0^\pi [t_n'(\theta)]^{2p} \sin^k\theta d\theta \le \frac{2p-1}{2p} n^2 \int_0^\pi [t_n'(\theta)]^{2p-2} \sin^k\theta d\theta + \frac{k\pi}{2p} n^{2p-1}$$

The above is best possible in the following manner.

$$\lim_{n\to\infty} \frac{1}{n^{2p}} \int_0^\pi [t_n'(\theta)]^{2p} \sin^k\theta d\theta = \lim_{n\to\infty} \frac{1}{n^{2p-2}} \left(\frac{2p-1}{2p}\right) \int_0^\pi [t_n'(\theta)]^{2p-2} \sin^k\theta d\theta$$

when $t_n(\theta) = \cos n(\theta - \alpha)$.

Proof

Define

(5.2.20) $$I_1 = \int_0^\pi [t_n'(\theta)]^{2p} \sin^k\theta d\theta.$$

Clearly, we may write

(5.2.21) $I_1 = I_2 + I_3$,

where

$$I_2 = \int_0^\pi [t_n'(\theta)]^{2p-2} \left[(t_n'(\theta))^2 - t_n(\theta)\, t_n''(\theta)\right] \sin^k\theta d\theta$$

and

$$I_3 = \int_o^\pi t_n''(\theta)\, t_n(\theta)\, \left[t_n'(\theta)\right]^{2p-2} \sin^k\theta\, d\theta$$

From integration by parts,

$$\begin{aligned} I_3 = -\int_o^\pi t_n'(\theta) \Big\{ &\left[t_n'(\theta)\right]^{2p-1} \sin^k\theta \\ &+ (2p-2)\, t_n''(\theta)\, t_n(\theta) \left[t_n'(\theta)\right]^{2p-3} \sin^k\theta \\ &+ k t_n(\theta) \left[t_n'(\theta)\right]^{2p-2} \sin^{k-1}\theta \cos\theta \Big\}\, d\theta \end{aligned} \tag{5.2.22}$$

From (5.2.22), we have

$$I_3 = -I_1 - (2p-2)\, I_3 + E, \tag{5.2.23}$$

where

$$E = -k \int_o^\pi t_n(\theta) \left[t_n'(\theta)\right]^{2p-1} \sin^{k-1}\theta \cos\theta\, d\theta .$$

Hence,

$$I_3 = \frac{-1}{2p-1} I_1 + \frac{1}{2p-1} E. \tag{5.2.24}$$

From (5.2.21) and (5.2.24), we have

$$\frac{2p}{2p-1} I_1 = I_2 + \frac{1}{2p-1} E. \tag{5.2.25}$$

Further,

$$\begin{aligned} \int_o^\pi \left[t_n'(\theta)\right]^{2p} \sin^k\theta\, d\theta = \frac{2p-1}{2p} \int_o^\pi &\left[t_n'(\theta)\right]^{2p-2} \\ &\cdot \left[\left(t_n'(\theta)\right)^2 - t_n(\theta)\, t_n''(\theta) \right] \sin^k\theta\, d\theta \\ &- \frac{k}{2p} \int_o^\pi t_n(\theta) \left[t_n'(\theta)\right]^{2p-1} \sin^{k-1}\theta \cos\theta\, d\theta \end{aligned} \tag{5.2.26}$$

We now state the following inequality.

(5.2.27) $\left|\left[t_n'(\theta)\right]^2 - t_n''(\theta)\, t_n(\theta)\right| \le n^2$ for $|t_n(\theta)| \le 1$.

Equality for every θ if and only if $t_n(\theta) = \cos n(\theta - \alpha)$. We now apply (5.2.27) to the first and Bernstein's Inequality to the second term on the right hand side of (5.2.26). We obtain

(5.2.28) $$\int_0^\pi \left[t_n'(\theta)\right]^{2p} \sin^k\theta\, d\theta \le \frac{2p-1}{2p}\, n^2 \int_0^\pi \left[t_n'(\theta)\right]^{2p-2} \sin^k\theta\, d\theta + \frac{k\pi}{2p}\, n^{2p-1}.$$

The lemma follows.

Proof of Theorems

We now present the proofs of Theorems 5.1 - 5.3.

Proof of Theorem 5.1

We let $p_{n+1}(x)$ be any real algebraic polynomial of degree n + 1 such that $|p_{n+1}(x)| \le (1 - x^2)^{\frac{1}{2}}$, for $-1 \le x \le 1$.

Now, we write

(5.3.1) $$p_{n+1}(x) = (1 - x^2)\, q_{n-1}(x)$$

where $q_{n-1}(x)$ is a real algebraic polynomial of degree n - 1. Further we have,

(5.3.2) $$|q_{n-1}(x)| \le (1 - x^2)^{-\frac{1}{2}}, \text{ for } -1 < x < 1.$$

Through repeated differentiation of (5.3.1), we obtain

(5.3.3) $$p_{n+1}^{(k)}(x) = (1 - x^2)\, q_{n-1}^{(k)}(x) - 2kx q_{n-1}^{(k-1)}(x) - (k-1)k q_{n-1}^{(k-2)}(x),$$

for k = 0, 1, 2, ...

From (5.3.3), we have

$$(5.3.4) \qquad \int_{-1}^{1} \left[p_{n+1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-3}{2}} dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

where

$$I_1 = \int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k+1}{2}} dx,$$

$$(5.3.5) \qquad I_2 = 4k^2 \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 x^2 (1 - x^2)^{\frac{2k-3}{2}} dx,$$

$$I_3 = (k-1)^2 k^2 \int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2 (1 - x^2)^{\frac{2k-3}{2}} dx,$$

and

$$I_4 = -4k \int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\, q_{n-1}^{(k-1)}(x)\right] x (1 - x^2)^{\frac{2k-1}{2}} dx.$$

Upon integration by parts

$$(5.3.6) \qquad \begin{aligned} I_4 &= 2k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1 - x^2)^{\frac{2k-3}{2}} (1 - 2kx^2)\, dx, \\ I_5 &= 4(k-1)k^2 \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\, q_{n-1}^{(k-2)}(x)\right] x (1 - x^2)^{\frac{2k-3}{2}} dx \end{aligned}$$

Similarly, we obtain

$$I_5 = -2\,(k-1)\,k^2 \int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2$$

(5.3.7)

$$\cdot\,(1-x^2)^{\frac{2k-5}{2}}\,[1-2\,(k-1)\,x^2]\,dx$$

Next,

$$I_6 = -2\,(k-1)\,k \int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\,q_{n-1}^{(k-2)}(x)\right](1-x^2)^{\frac{2k-1}{2}}dx$$

$$= 2\,(k-1)\,k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}dx$$

$$-\,2\,(2k-1)\,(k-1)\,k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\,q_{n-1}^{(k-2)}(x)\right] x\,(1-x^2)^{\frac{2k-3}{2}}dx$$

Note that

$$\int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\,q_{n-1}^{(k-2)}(x)\right] x\,(1-x^2)^{\frac{2k-3}{2}}dx$$

$$= -\frac{1}{2}\int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}}\,[1-2\,(k-1)\,x^2]\,dx$$

Hence,

$$I_6 = 2\,(k-1)\,k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}}dx$$

(5.3.8)

$$+\,(2k-1)\,(k-1)\,k \int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2$$

$$\cdot\,(1-x^2)^{\frac{2k-5}{2}}\,[1-2\,(k-1)\,x^2]\,dx$$

From (5.3.4) - (5.3.8), we obtain

$$\int_{-1}^{1} \left[p_{n+1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx = \int_{-1}^{1} \left[q_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}} dx$$

$$+ 2(k-1)k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx$$

(5.3.9)
$$+ 2k \int_{-1}^{1} \left[q_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ (k-2)(k-1)^2 k \int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ (2k-3)(k-1)k \int_{-1}^{1} \left[q_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx \, .$$

Finally, we use Lemmas 5.2 and 5.3 along with (5.3.2) and (5.3.9) to get

$$\int_{-1}^{1} \left[p_{n+1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx \le \int_{-1}^{1} \left[u_{n-1}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k+1}{2}} dx$$

$$+ 2(k-1)k \int_{-1}^{1} \left[u_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx$$

$$+ 2k \int_{-1}^{1} \left[u_{n-1}^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ (k-2)(k-1)^2 k \int_{-1}^{1} \left[u_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ (2k-3)(k-1)k \int_{-1}^{1} \left[u_{n-1}^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx$$

$$= \int_{-1}^{1} \left[f_o^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

where $f_o(x) = (1 - x^2)u_{n-1}(x)$.

Equality iff $p_{n+1}(x) = \pm f_o(x)$. Proof of Theorem 5.1 is complete.

Proof of Theorem 5.2

We let $p_{n+2}(x)$ be any real algebraic polynomial of degree n + 2 such that $|p_{n+2}(x)| \leq (1 - x^2)$, for $-1 \leq x \leq 1$.

Now we write,

(5.3.10) $\quad p_{n+2}(x) = (1 - x^2)\, q_n(x)$

where $q_n(x)$ is a real algebraic polynomial of degree n.

Further, we have

(5.3.11) $\quad |q_n(x)| \leq 1$, for $-1 \leq x \leq 1$.

Through repeated differentiation of (5.3.10), we get

(5.3.12) $\quad p_{n+2}^{(k)}(x) = (1 - x^2)\, q_n^{(k)}(x) - 2kx q_n^{(k-1)}(x) - (k - 1)k\, q_n^{(k-2)}(x)$

for k = 0, 1, 2, ...

From (5.3.12), we have

(5.3.13) $$\int_{-1}^{1} \left[p_{n+2}^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-5}{2}} dx = I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

where

$$I_1 = \int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1 - x^2)^{\frac{2k-1}{2}} dx,$$

(5.3.14) $$I_2 = 4k^2 \int_{-1}^{1} \left[q_n^{(k-1)}(x)\right]^2 x^2 (1 - x^2)^{\frac{2k-5}{2}} dx,$$

$$I_3 = (k - 1)^2 k^2 \int_{-1}^{1} \left[q_n^{(k-2)}(x)\right]^2 (1 - x^2)^{\frac{2k-5}{2}} dx,$$

and

$$I_4 = -4k \int_{-1}^{1} \left[q_n^{(k)}(x)\, q_n^{(k-1)}(x)\right] x\,(1 - x^2)^{\frac{2k-3}{2}}\, dx.$$

Using integration by parts,

(5.3.15)

$$I_4 = 2k \int_{-1}^{1} \left[q_n^{(k-1)}(x)\right]^2 (1 - x^2)^{\frac{2k-5}{2}} \left[1 - 2(k-1)x^2\right] dx$$

$$I_5 = 4(k-1)k^2 \int_{-1}^{1} \left[q_n^{(k-2)}(x)\, q_n^{(k-1)}(x)\right] x\,(1 - x^2)^{\frac{2k-5}{2}}\, dx$$

Similarly, we obtain

(5.3.16)
$$I_5 = -2(k-1)k^2 \int_{-1}^{1} \left[q_n^{(k-2)}(x)\right]^2 \cdot (1 - x^2)^{\frac{2k-7}{2}} \left[1 - 2(k-2)x^2\right] dx$$

Next,

$$I_6 = -2(k-1)k \int_{-1}^{1} \left[q_n^{(k-2)}(x)\, q_n^{(k)}(x)\right] (1 - x^2)^{\frac{2k-3}{2}}\, dx$$

$$= 2(k-1)k \int_{-1}^{1} \left[q_n^{(k-1)}(x)\right]^2 (1 - x^2)^{\frac{2k-3}{2}}\, dx$$

$$- 2(2k-3)(k-1)k \int_{-1}^{1} \left[q_n^{(k-1)}(x)\, q_n^{(k-2)}(x)\right] x\,(1 - x^2)^{\frac{2k-5}{2}}\, dx$$

Observe that

$$\int_{-1}^{1} \left[q_n^{(k-1)}(x)\, q_n^{(k-2)}(x)\right] x\,(1-x^2)^{\frac{2k-5}{2}} dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \left[q_n^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-7}{2}} \left[1 - 2(k-2)x^2\right] dx \,.$$

Hence,

(5.3.17)

$$I_6 = 2(k-1)k \int_{-1}^{1} \left[q_n^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ (2k-3)(k-1)k \int_{-1}^{1} \left[q_n^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-7}{2}} \left[1 - 2(k-2)x^2\right] dx \,.$$

From (5.3.13) - (5.3.17), we may write

(5.3.18)

$$\int_{-1}^{1} \left[p_{n+2}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx = \int_{-1}^{1} \left[q_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx$$

$$+ 2(k-3)k \int_{-1}^{1} \left[q_n^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ 6k \int_{-1}^{1} \left[q_n^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx$$

$$+ (k-4)(k-3)(k-1)k \int_{-1}^{1} \left[q_n^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx$$

$$+ 3(2k-5)(k-1)k \int_{-1}^{1} \left[q_n^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-7}{2}} dx$$

Finally, we use Lemmas 5.5 and 5.6 along with (5.3.11) and (5.3.18) to obtain

$$\int_{-1}^{1} \left[p_{n+2}^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx \le \int_{-1}^{1} \left[T_n^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-1}{2}} dx$$

$$+ 2(k-3)k \int_{-1}^{1} \left[T_n^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-3}{2}} dx$$

$$+ 6k \int_{-1}^{1} \left[T_n^{(k-1)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx$$

$$+ (k-4)(k-3)(k-1)k \int_{-1}^{1} \left[T_n^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx$$

$$+ 3(2k-5)(k-1)k \int_{-1}^{1} \left[T_n^{(k-2)}(x)\right]^2 (1-x^2)^{\frac{2k-7}{2}} dx$$

$$= \int_{-1}^{1} \left[f_1^{(k)}(x)\right]^2 (1-x^2)^{\frac{2k-5}{2}} dx .$$

where $f_1(x) = (1 - x^2)\, T_n(x)$. Equality iff $p_{n+2} = \pm f_1(x)$.

<u>Proof of Theorem 5.3</u>

Define $f_o(x)$ such that $P_{n+2}(x) = (1 - x^2) f_o(x)$. Then $f_o(x)$ is a real algebraic polynomial of degree n such that $|f_o(x)| \le 1$ for $-1 \le x \le 1$. We now define

(5.3.19) $\quad t_n(\theta) = f_o(\cos\theta)$ for $o \le \theta \le \pi$.

Then $t_n(\theta)$ is a trigonometric polynomial of order n such that

(5.3.20) $\quad |t_n(\theta)| \le 1$ for $o \le \theta \le \pi$.

From the binomial theorem and Bernstein's Inequality,

$$\int_{-1}^{1} |P_{n-2}'(x)|^{2p}\,dx = \int_{o}^{\pi} \left[\sin\theta\, t_n'(\theta) + 2\cos\theta\, t_n(\theta)\right]^{2p} \sin\theta\, d\theta$$

$$= \binom{2p}{0} \int_{o}^{\pi} \left[\sin\theta\, t_n'(\theta)\right]^{2p} \sin\theta\, d\theta + \binom{2p}{1}$$

(5.3.21)
$$\cdot \int_{o}^{\pi} \left[\sin\theta\, t_n'(\theta)\right]^{2p-1} [2\cos\theta\, t_n(\theta)]^{1} \sin\theta\, d\theta$$

$$+ \ldots + \binom{2p}{2p} \int_{o}^{\pi} [2\cos\theta\, t_n(\theta)]^{2p} \sin\theta\, d\theta \le \int_{o}^{\pi} \left[t_n'(\theta)\right]^{2p} \sin^{2p+1}\theta\, d\theta$$

$$+ \binom{2p}{1} (2)^{1}\, \pi\, n^{2p-1} + \binom{2p}{2} (2)^{2}\, \pi\, n^{2p-2} + \ldots + \binom{2p}{2p} (2)^{2p}\, \pi\, n^{o} .$$

After repeated applications of Lemma 5.7, we have

$$\int_{o}^{\pi} \left[t_n'(\theta)\right]^{2p} \sin^{2p+1}\theta\, d\theta$$

(5.3.22)
$$\le \left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{2p-1}{2p}\right) \int_{o}^{\pi} \sin^{2p+1}\theta\, d\theta\right] n^{2p}$$

$$+ \frac{(2p+1)\pi}{2} \left[\frac{n^{2p-1}}{p} + \frac{n^{2p-3}}{p-1} + \ldots + \frac{n^3}{2} + \frac{n}{1}\right] .$$

We conclude that

$$\int_{-1}^{1} |P_{n+2}'(x)|^{2p}\,dx \le \left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{2p-1}{2p}\right) \int_{o}^{\pi} \sin^{2p+1}\theta\, d\theta\right] n^{2p}$$

(5.3.23)
$$+ \frac{(2p+1)\,\pi}{2} \left[\frac{n^{2p-1}}{p} + \frac{n^{2p-3}}{p-1} + \ldots + \frac{n^3}{2} + \frac{n}{1}\right]$$

$$+ \binom{2p}{1} 2\pi n^{2p-1} + \binom{2p}{2} 2^2\pi n^{2p-2} + \ldots + \binom{2p}{2p} 2^{2p}\pi n^{o} .$$

The theorem now follows.

CHAPTER 6
SUMMARY AND CONCLUSIONS

The problems addressed in this dissertation are taken from the vast field of approximation theory. The three areas considered are Birkhoff interpolation, lacunary spline interpolation, and Markov-type inequalities.

In Chapters Two and Three, we discussed the explicit representation and convergence properties for a Birkhoff interpolation process. In Chapter Two, we found the unique polynomial of minimal degree that takes function values on one set of knots while its second derivative takes the value zero on another set of knots. Specifically, function values are interpolated at the zeros of $(1 - x^2)P_{n-1}(x)$, and the second derivative values are prescribed to be zero at the zeros of $P'_{n-1}(x)$. Chapter Three gives a pointwise estimate for the error in using these polynomials to approximate a continuous function on the interval $[-1,1]$. The error is shown to be of order

$$\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right) \text{ for } -1 \le x \le 1.$$

We note that we have given a discrete interpolatory proof of the Teljakovskii Theorem (Theorem 1.5).

Chapter Four discussed (0,1,3) and (0,1,2,4) lacunary spline interpolation. Here, we allowed some of the data to be prescribed at the midpoints of the joints as well as at the joints of the spline. The splines constructed converge to an interpolated function with the optimal order of convergence for splines of their class. One of the (0,1,2,4) splines presented turns out to be a locally defined process.

In Chapter Five, we obtained various upper bounds for the derivative of polynomials under curved majorants in the L^p norm. Here, p is an even integer and the majorants considered are the circular and parabolic ones.

We conclude this section with some open problems inspired by the discussion presented in this work. In Chapter Three, one could improve the pointwise error estimate given in Theorem 3.1 by showing the result holds for the second modulus of continuity. This would provide a discrete, interpolatory proof of the DeVore Theorem (Theorem 1.6).

Another related problem would be to obtain the explicit forms and convergence properties for a similar Pál-type (0;2) interpolation process involving the zeros of the Tchebycheff polynomials of the first and second kind. Do such polynomials converge for the entire class of continuous functions? What sort of pointwise estimation of error is possible?

In Theorem 5.3, we present an upper bound for the first derivative of polynomials under the parabolic majorant in the L^p norm, where p is an even integer. Our upper bound is shown to be asymptotically sharp as discussed in (5.1.3). One could improve this result by giving an upper bound that is sharp for every n. In addition, one might ask what types of upper bounds are possible for higher derivatives.

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